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Micro foundations of multi-prize lottery contests: a perspective of noisy performance ranking

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Abstract This article proposes a multi-prize noisy-ranking contest model. Contestants are ranked in descending order based on their perceived performance, which is subject to random perturbation, and they are rewarded based on their ranks. Under plausible conditions, we establish that our noisy performance ranking model is stochastically equivalent to the family of multi-prize lottery contests built upon ratio-form contest success functions. We further establish the equivalence of our model to a contest model that ranks contestants by their best performance out of multiple independent attempts. These results therefore shed light on the micro-foundations of the popularly adopted lottery contest models. The "best-shot ranking rule" reveals a common thread that connects a broad class of seemingly disparate competitive activities (such as rent-seeking contests, patent races, research tournaments), and unifies them through a common performance evaluation mechanism.

1 Introduction

A wide class of competitive activities can be viewed as contests, in which all participants forfeit scarce resources to compete for a limited number of prizes. The prevalence

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of this phenomenon has spawned an enormous amount of economic literature, and a wealth of theoretical contest/tournament models. These models exhibit a variety of technical characteristics and have been applied in a wide range of contexts, including college admissions, influence politics, sports, war and conflict, and internal labor market competition.[1]

Central to formally modeling contests is a mechanism that picks the winners and distributes the prizes. In "imperfectly discriminatory contests" , the amount of effort exerted directly affects whether or not one wins, but measurement errors, subjective biases, and randomness in the production processes can also influence the outcome. The selection mechanism in this type of contest is conventionally represented by a *contest success function*, which maps contestants' effort entries into the likelihood of every contestant winning each prize.

Most of the existing literature focuses on winner-take-all contests. Perhaps the most-widely adopted approach is the lottery contest model, which assumes a ratio-form contest success function. The Tullock contest model is its most popular special case. In a winner-take-all lottery contest, the likelihood that a contestant i wins, P_i, is given by the ratio of the output of his/her effort to the total output supplied by the entire cohort, i.e., $P_i = g_i(x_i)/\sum_{j=1}^{n} g_j(x_j)$, where the output production function $g_i(x_i)$ is usually an increasing function of effort x_i.[2]

This framework provides an intuitive and tractable specification for the winning probability as a function of effort in winner-take-all imperfect discriminatory contests. However, a ratio-form contest success function does not directly apply to the widely observed multi-prize contests, where contestants vie for more than one prize.[3] To fill this gap, Clark and Riis (1996b, 1998a) introduce a clever "generalization" of the basic Tullock contest model that allows a block of prizes to be distributed. By adopting ratio-form success functions as its building block, this "*multiple-winner nested-contest model*" hypothetically conducts a series of conditionally independent (single-winner) "lotteries." Each lottery "draws" one prize recipient until all the prizes are given away.[4]

Thus far, the nested-contest model offers the most reasonable and convenient alternative for determining multiple prize recipients in imperfectly discriminatory contests.[5,6] Nevertheless, the nature of this (seemingly sequential) process deserves

1 See Konrad (2009) for a thorough survey of economic studies on contests.

2 Recent applications of ratio-form contest success functions can be seen in Wärneryd (2000), Yildirim (2005), Morgan and Várdy (2007) and Hann et al. (2008), among many others.

3 Examples include (a) firms setting aside a number of bonus packages to reward their top-performing workers; (b) employees competing to fill multiple vacancies that are higher in the organizational hierarchy; and (c) the awarding of gold, silver, and bronze medals to runners-up in sports competitions.

4 As a result, the conditional probability that a remaining contestant will be selected in the next "draw" is independent of the effort exerted by contestants selected in previous "draws."

5 Besides the studies conducted by Clark and Riis (1996b, 1998a), lottery contest models have been applied in multiple-winner settings by Amegashie (2000), Yates and Heckelman (2001), Szymanski and Valletti (2005) and Fu and Lu (2009a,b).

6 Another approach to modeling multiple-winner contests is the multiple-prize all-pay auction model. A handful of studies have contributed to this research agenda, including Barut and Kovenock (1998), Clark and Riis (1998b), Moldovanu and Sela (2001), Moldovanu et al. (2007), Siegel (2009).

further exploration, and its intrinsic connection to the winner-take-all lottery contest model has yet to be investigated. The economic activities underlying each single "lottery" , as well as the entire (seemingly sequential) selection process, remain in a "black box." In this article, we set out to address these issues.

Konrad (2009) points out in his thorough survey of economic studies on contests that a contest can be naturally regarded as a competitive event where contestants expend costly efforts to "get ahead of their rivals." Based on this notion, a contest requires contestants to be (at least partially) ordered based on a ranking system. We then raise the following question: *Is there a ranking system that underpins the winner selection mechanism modeled by these lottery contests?*

We show that a unique ranking system does exist. We propose a multi-prize contest model that selects prize recipients through a noisy ranking of contestants. A fixed number of economic agents (contestants) produce their outputs from their inputs (effort). Following Lazear and Rosen (1981), one's observed output is the sum of a deterministic component (a strictly increasing function of effort) and a noise term that could arise from numerous sources, e.g., perturbation in production and performance measurement error. These contestants are ranked by their observed output in descending order[7]: the higher the observed output, the better a contestant's rank. As a result, given a set of effort entries, and any (simultaneous) realization of noise terms, a complete ranking arises. Each agent is awarded a prize of his/her rank.[8]

Our framework borrows its technical form directly from the consumers' discrete choice econometric model (also known as McFadden's general extreme value model). Formal statistical analysis (which is laid out in Sect. 2.3) reveals the economic nature of our model in a contest setting. When the noise term follows a Type-I extreme value (maximum) distribution, the noisy-ranking contest is underpinned by a "favorable performance (FP) ranking" rule. In other words, the ranking rule essentially honors the most FP of each contestant, i.e., "the best shot" of each contestant, across multiple independent attempts. The performance of each attempt could follow any distribution. This allows us to interpret our framework as a "best-shot" contest.

We establish that our "best-shot" contest, underpinned by FP ranking, uniquely generates winning probabilities that are identical to those of a multiple-winner nested contest model (Clark and Riis 1996b, 1998a). For any given effort entries and production functions, the ex ante likelihood of every possible prize distribution outcome perfectly coincides with that in the multiple-winner nested contest model.[9] The hypothetical sequential-lottery procedure of this nested contest model virtually reflects a

[7] An analogous scenario is the standard moral hazard setting, where an employer cannot observe or verify the effort supplied by contestants (employees). Hence, he ranks the perceived performance of contestants to determine their compensation and other rewards.

[8] According to Clark and Riis (1998a) and Fu and Lu (2009b), this prize allocation rule maximizes the amount of individual effort in a multi-prize lottery contest (multiple-winner nested contest). As this article will establish the stochastic equivalence between our noisy-ranking model and a multi-prize lottery contest, with this prize allocation rule also maximizing the expected output in our framework. This type of rank-ordered prize distribution rule has also been discussed in studies by Glazer and Hassin (1988), Barut and Kovenock (1998), Moldovanu and Sela (2001), etc.

[9] The ex ante likelihood that a contestant is ordered on the l-th rank is equivalent to the probability that a contestant is selected for the l-th draw in a multiple-winner nested contest.

statistical property of its underlying (simultaneous) FP ranking system. As a winner-take-all lottery is a special case of the nested contest model (where only the first draw is of interest), the winner-take-all lottery contest is thus integrated with the multiple-prize nested contest model through the (unique) FP ranking.

Our results provide statistical micro-foundations for lottery contest models. The FP ranking underlying our ranking model illuminates the competitive activities encapsulated in ratio-form success functions. We show that an evaluation rule that honors contestants' "best shots" can be explicitly witnessed in many real-life competitive events and conforms to a natural regularity. The FP ranking uncovered in our analysis further allows lottery contest models to be connected to other modeling approaches and to unify a wide variety of observationally detached competitive activities. We demonstrate the roles played by FP ranking rule in these diverse contexts. The analysis also implies a limit on the scope of this relationship. We offer one example of a "weakest link" contest, where a "best-shot" evaluation mechanism (FP ranking) is missing, to further illuminate the nature of lottery contests.

A handful of articles have probed the micro-foundations of *winner-take-all* contest success functions. Skaperdas (1996) shows that the ratio-form contest function is the only alternative that satisfies a number of axiomatic properties. This axiomatic foundation, as pointed out by Skaperdas (1996) and Clark and Riis (1996a, 1997), alludes to the hidden connection between the contest models, the probabilistic choice models (Luce and Suppes 1965), and the discrete choice econometric models (McFadden 1973, 1974).[10] In particular, Clark and Riis (1996a) explicitly point out the equivalence of a random choice model and lottery contests in the winner-take-all case.[11,12] Our article complements this literature in two aspects. First, this is the *first* attempt in the literature to provide micro-foundations from a noisy-ranking perspective for the generalized lottery contest models that allow for *multiple* prizes. Our analysis uncovers that the nested multiple-winner contest model (Clark and Riis 1996b) is naturally integrated with the standard lottery contest framework. Second, our study reveals a plausible economic interpretation ("best-shot" contests) for the popularly assumed ratio-form success functions, which allows us to link together a variety of models from diverse contexts. Hence, this article is also related to the literature that bridges different contest modeling approaches. For instance, Baye and Hoppe (2003) reveal that Research Tournament Models (Fullerton and McAfee 1999), Patent Race Models (Dasgupta and Stiglitz 1980) and winner-take-all Tullock Contest Models are strategically equivalent. Our article explores the (unique) statistical and economic foundation that underpins the equivalence among these diverse frameworks. In addition, our analysis focuses on a more general environment which allows for multiple prizes.

[10] Assuming unmeasurable psychological factors, this literature investigates randomized choices of decision makers (consumers) that result from a stochastic ranking. Among others, McFadden (1973, 1974) has demonstrated the econometric implementation of modeling revealed choice among discrete alternatives while adopting a probabilistic choice model.

[11] This stochastic property is rediscovered by Jia (2008) in the setting of single-prize contests.

[12] Clark and Riis (1996a) interpret the noise in the evaluation process as bias in observation, while we interpret it as perturbation in production process.

The remainder of the paper is organized as follows. In Sect. 2, the model is set up, the analysis is completed, and the implications of this model are briefly discussed. Section 3 reinforces the argument by presenting the "dual" problem to our benchmark model and an "antithesis." Section 4 provides some concluding remarks.

2 A multi-prize noisy-ranking contest model

2.1 Setup

$I \geq 2$ contestants, indexed by $i \in \mathbf{I} \triangleq \{1, 2, \ldots, I\}$, simultaneously commit to their effort $\mathbf{x} = (x_1, \ldots x_I)$, to compete for $L \in \{1, 2, \ldots, I\}$ prizes. A contestant's effort allows him to produce a perceivable output (y_i), which is subject to random perturbation. Contestants are evaluated through these noisy signals of their performance (y_is). It should be pointed out that a tournament model with noisy performance of contestants can be traced back to the original work of Lazear and Rosen (1981).

Based on the discrete choice framework of McFadden (1973, 1974), we assume that the noisy signal (y_i) is described by

$$\log y_i = \log g_i(x_i) + \varepsilon_i, \quad \forall i \in \mathbf{I}, \tag{1}$$

where the deterministic and strictly increasing function $g_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ measures the output of contestant i's effort x_i,[13] and the additive noise term $\varepsilon_i \in (-\infty, +\infty)$ reflects the randomness in the production process or the imperfection of the measurement and evaluation process. $g_i(\cdot)$ is named as the production function of contestant i. Define $\mathbf{g} \triangleq \{g_i(\cdot), i \in \mathbf{I}\}$, which denotes the set of production technologies. The idiosyncratic noises $\varepsilon \triangleq \{\varepsilon_i(\cdot), i \in \mathbf{I}\}$ are independently and identically distributed with zero means.

The L prizes are ordered by their values, with $V_1 \geq V_2 \geq \cdots \geq V_L$. Each contestant is eligible for one prize at the most. These contestants are ranked based on their perceivable performance (i.e., $\log y_i$) in descending order. Prizes are allocated among contestants based on their ranks, given the availability of the prizes. That is, the contestant who contributes the highest output y_i receives V_1, the contestant who contributes the second highest output then receives V_2, and so on, until all the prizes are given away.

When $L = 1$, the model degenerates into a winner-take-all contest, with the top-ranked contestant being the only winner. When $L \geq 2$, a multi-prize contest follows, which requires a more complete ranking among contestants to implement its prize distribution rule. For any given effort entries $\mathbf{x}$, a complete ranking among contestants immediately results from any realization of the noise terms ε. We assume that ties are broken randomly and fairly. The probability of a contestant i winning a prize V_l is simply given by the probability that he/she is ranked at the l-th position. This setup therefore embraces the notion that a contest is a competitive event where contestants compete to "get ahead of others" (Konrad 2009).

[13] Define $\log g_i(x_i) = -\infty$ if $g_i(x_i) = 0$.

While this model imposes virtually no restrictions on the technology $g_i(\cdot)$ and the number of prizes L, it follows McFadden (1973, 1974) in assuming the random component ε_i to be drawn from a type I extreme-value (maximum) distribution. The cumulative distribution function of ε_i is

$$F(\varepsilon_i) = e^{-e^{-\varepsilon_i}}, \quad \varepsilon_i \in (-\infty, +\infty), \forall i \in \mathbf{I}, \tag{2}$$

and the density function is

$$f(\varepsilon_i) = e^{-\varepsilon_i - e^{-\varepsilon_i}}, \quad \varepsilon_i \in (-\infty, +\infty), \forall i \in \mathbf{I}. \tag{3}$$

The mean and variance of ε_i are given by $\gamma(\approx 0.5772)$ and $\frac{1}{6}\pi^2$, respectively, where γ is the Euler–Mascheroni constant.

The performance evaluation mechanism underlying this formulation will be discussed in Sect. 2.3, which reveals the economic implications of this seemingly peculiar distribution. Further, it should be noted that our model can be set up in alternative but technically equivalent ways. For instance, the additive-noise ranking model (1) is equivalent to a multiplicative-noise ranking model;

$$y_i = g_i(x_i)\tilde{\varepsilon}_i, \quad \forall i \in \mathbf{I}, \tag{4}$$

where the noise term $\tilde{\varepsilon}_i$ is defined as $\tilde{\varepsilon}_i \triangleq \exp \varepsilon_i$. However, the current setup, i.e., model (1), allows for the most handy subsequent analysis, which is to be executed in Sect. 2.2. When ε_i follows a type I extreme-value (maximum) distribution, $\tilde{\varepsilon}_i \triangleq \exp \varepsilon_i$ must follow a Weibull (maximum) distribution. As will be revealed in Sects. 2.3 and 3, the two types of models have the same economic implications.

In the subsequent analysis, we first show that this noisy-ranking model is stochastically equivalent to the family of lottery contests (winner-take-all lottery contests and multiple-winner nested contests). We then analytically explore the economic nature of this model.

2.2 The equivalence to lottery contests

Given effort entry $\mathbf{x}$, contestant i is ranked ahead of contestant j, if and only if

$$\begin{aligned} \log g_i(x_i) + \varepsilon_i \ &\geq \ \log g_j(x_j) + \varepsilon_j \\ &\Leftrightarrow \varepsilon_j \leq \varepsilon_i + \log \frac{g_i(x_i)}{g_j(x_j)}. \end{aligned}$$

Contestant i will be ranked the highest if and only if

$$\varepsilon_j \leq \varepsilon_i + \log \frac{g_i(x_i)}{g_j(x_j)}, \quad \forall j \in \mathbf{I} \backslash \{i\}.$$

McFadden (1973, 1974) characterizes the probability distribution of the top-ranked choice. The result can be adapted to our contest setting.

Lemma 1 (McFadden 1973, 1974) *For any given* $\mathbf{x} \geq 0$ *such that* $\sum_{j\in\mathbf{I}} g_j(x_j) > 0$, *the ex ante likelihood that a contestant* i *achieves the top rank is*

$$p(i|\mathbf{x}) = \frac{g_i(x_i)}{\sum_{j\in\mathbf{I}} g_j(x_j)}, \quad \forall i \in \mathbf{I}. \tag{5}$$

The proof is omitted as it is available from McFadden (1973, 1974). By Lemma 1, the probability that a contestant will be ranked the highest can be expressed as the ratio of his/her deterministic output $g_i(x_i)$ to the sum $\sum_{j\in\mathbf{I}} g_j(x_j)$. This winning probability coincides with the popularly assumed ratio-form contest success function in winner-take-all lottery contests.

When $L \geq 2$ and $I \geq 3$, the model is a multi-prize contest.[14] To fully describe a multi-prize contest, the probability of each contestant winning each prize has to be completely characterized. For this purpose, we now characterize the probabilities of all possible complete rankings for a given set of effort entries $\mathbf{x}$.

Let the sequence $\{i_k\}_{k=1}^{I}$ denote a complete ranking of the I contestants, where i_k is the index of the k-th ranked contestant. We obtain the following result:

Lemma 2 *For any given effort entries* $\mathbf{x} \geq 0$ *such that* $g_i(x_i) > 0, \forall i \in \mathbf{I}$, *the ex ante likelihood of any complete ranking outcome* $\{i_k\}_{k=1}^{I}$ *can be expressed as*

$$p(\{i_k\}_{k=1}^{I}) = \Pi_{k=1}^{I} \frac{g_{i_k}(x_{i_k})}{\sum_{k'=k}^{I} g_{i_{k'}}(x_{i_{k'}})}. \tag{6}$$

Proof See Appendix A.1. □

Lemma 2 states that the ex ante likelihood of a complete ranking can be expressed as the cumulative product of all the terms of the series $\{g_{i_k}(x_{i_k})/\sum_{k'=k}^{I} g_{i_{k'}}(x_{i_{k'}})\}_{k=1}^{I}$. The term $g_{i_k}(x_{i_k})/\sum_{k'=k}^{I} g_{i_{k'}}(x_{i_{k'}})$, as shown in the Appendix A.1, gives the probability of a contestant i_k being ranked in the k-th place, conditional on his/her rank falling below $k-1$.[15]

The L prizes are awarded to the L contestants who contribute the highest y_is, correspondingly, based on their ranks. Hence, a prize distribution outcome is represented by the subsequence $\{i_k\}_{k=1}^{L}$ of $\{i_k\}_{k=1}^{I}$, where i_k denotes the index of the contestant who is ranked at the k-th position and receives V_k. The probability of a prize distribution outcome $\{i_k\}_{k=1}^{L}$ is given as follows in light of Lemma 2.

14 A multi-prize contest is sensible only if $I \geq 3$. When $I = 2$ and $L = 2$, a two-prize contest with (V_1, V_2) is strategically equivalent to a single-prize contest with $V_1 - V_2$ if there is no income effect in contestants' utility functions.

15 This property was first non-constructively proposed by Luce and Suppes (1965) as a hypothetical decision rule. It was first used in the econometrics literature by Beggs et al. (1981). To our knowledge, it has not been applied in the multi-prize contest literature.

Theorem 1 *For any given effort entries* $\mathbf{x} \geq 0$ *such that* $g_i(x_i) > 0, \forall i \in \mathbf{I}$, *the ex ante likelihood of any prize distribution outcome* $\{i_k\}_{k=1}^{L}$, $L \geq 1$ *can be expressed as*

$$p(\{i_k\}_{k=1}^{L}) = \Pi_{k=1}^{L} \frac{g_{i_k}(x_{i_k})}{\sum_{k'=k}^{I} g_{i_{k'}}(x_{i_{k'}})}. \tag{7}$$

To complete the model, we need to discuss the remaining cases that Theorem 1 does not cover, where some $g_i(x_i) = 0$ so that the outputs y_i are zero for some contestants. Since prizes are first allocated to the contestants with higher outputs and ties are randomly and fairly broken in our noisy-ranking model, these contestants will be ranked among the bottom-most in a random manner.

We now elaborate upon the connection between our noisy-ranking contest model and the multiple-winner nested contest Clark and Riis (1996b, 1998a). Clark and Riis (1996b, 1998a) extend winner-take-all Tullock contests to allow for a block of prizes to be allocated among contestants. Contestants simultaneously submit their one-shot effort entries $\mathbf{x}$. The recipient of each prize is selected through a "lottery" among all the remaining eligible candidates, with each "lottery" being represented by a ratio-form contest success function. A single prize recipient is drawn in each lottery draw. As each contestant is eligible for one prize at the most, the recipient of a prize is immediately removed from the pool of candidates who are eligible for the next draw. This procedure is repeated until all the prizes are given away. Let Ω_m denote the index set of all the remaining contestants for the m-th draw for the m-th prize V_m. Then the probability that any contestant $j \in \Omega_m$ wins a prize V_m is equal to $\frac{f_j(x_j)}{\sum_{i \in \Omega_m} f_i(x_i)}$ if $\sum_{i \in \Omega_m} f_i(x_i) > 0$. Here, $f_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is the output function of contestant i in the contest, and is assumed to be strictly increasing with effort outlay x_i. To the extent that $\sum_{i \in \Omega_m} f_i(x_i) = 0$, i.e., $f_i(x_i) = 0, \forall i \in \Omega_m$, prizes are randomly given away. Thus, the prize distribution outcome of this nested contest is determined by a series of M independent lotteries if M prizes are available. For a given set of effort outlays $\mathbf{x}$, the likelihood of any prize distribution outcome is simply the same as (7). A *stochastic equivalence* is therefore established between the nested model and our ranking model in the sense that they generate the same probability for any prize allocation outcome.

The nested model is implemented (literally) through a sequential lottery process. In our noisy-ranking model, the ranks of all contestants are nevertheless determined simultaneously. Our analysis thus sheds light on the microeconomic underpinning of this (seemingly sequential) multi-prize lottery contest model from the perspective of (simultaneous) noisy performance ranking.

An intriguing question naturally arises: does another distribution of the noise term ε_i exist which can deliver the probability distribution given by (7)? The answer is in the affirmative when $I = 2$. Hirshlefer and Riley (1992) provide one such example in winner-take-all contests, which are described in Sect. 3.2. However, the answer is negative for $I > 2$.

Let $C(\mathbf{I}, \mathbf{g}(\cdot), \mathbf{V})$ denote a multiple-winner nested Tullock contest with contestants $\mathbf{I}$, output functions $\mathbf{g}(\cdot)$ and prizes $\mathbf{V}$: each contestant i is endowed with an output production technology $g_i(x_i)$; while the vector $\mathbf{V} = (V_1, \ldots, V_L)$ represents the ordered set of L prizes with $V_1 \geq V_2 \geq \cdots \geq V_L$. We obtain the following:

Theorem 2 *When $I \geq 3$ and $L \geq 1$, the benchmark noisy-ranking model* (1) *is equivalent to the generalized multiple-winner nested contest model* $C(\mathbf{I}, \mathbf{g}(\cdot), \mathbf{V})$ *if and only if* ε_i *follows a type I extreme-value (maximum) distribution with c.d.f. of* $F(\varepsilon_i) = e^{-e^{-(\varepsilon_i+b)}}$, *with* $\varepsilon_i \in (-\infty, +\infty)$ *and* $b \in \mathbf{R}$.

Proof See Appendix A.2. □

Theorem 2 establishes the unique stochastic equivalence of the noisy-ranking contest model with the family of lottery contest models (winner-take-all and multi-prize).[16] Theorem 2 "uncovers" the stochastic nature of this lottery contest framework. The sequential lottery process conveniently reflects a statistical property of a hidden (simultaneous) noisy-ranking rule, while the probabilistic prize distribution rule (7) does not rely on a sequentially implemented selection mechanism.

The nested contest is reduced to a standard winner-take-all lottery contest when only one prize is available.[17] From the perspective of noisy rankings, it is clear that the "multiple-winner nested contest model" and winner-take-all lottery contest are integrated into a unified framework through a unique underlying ranking system.

2.3 Favorable performance ranking and "Best-Shot" contests

We now demonstrate that a lottery contest model can be approximated by a *"Best-Shot" Contest* model, or *Favorable Performance (FP) Ranking* Model, in which each contestant is ranked against others based on his/her most *favorable* performance among multiple independent attempts. The results in this subsection will unveil the micro-foundations and economic implications of our benchmark noisy-ranking model and lottery contest models.

We start with the noisy-ranking model (2), which is a direct variant of our benchmark model (1). In this model, contestants are ranked in descending order by the realization of y_is. As mentioned earlier, the noise term $\tilde{\varepsilon}_i$ is defined as $\tilde{\varepsilon}_i \triangleq \exp \varepsilon_i$. It is straightforward to verify that y_i follows a Weibull (maximum) distribution with the cumulative distribution function $\Pr(y_i \leq y) = e^{-\frac{g_i(x_i)}{y}}$.

Consider a uniform increasing transformation of variables y_i: $\zeta_i = F_0^{-1}\left(e^{-\frac{1}{y_i}}\right)$, $i = 1, 2, \ldots, I$, where $F_0(\cdot) : \mathbb{R}_+ \rightarrow [0, 1]$ is an arbitrary continuous and strictly increasing function. The random variable ζ_i, $i = 1, 2, \ldots, I$, is also independently distributed. Because ζ_i is a monotonic transformation of y_i, ranking these contestants by y_is is equivalent to ranking them by ζ_is. The cumulative distribution function of ζ_i is given by

[16] Clark and Riis (1998a), as well as Fu and Lu (2009a), have provided a complete solution for the multiple-winner nested contests when contestants are symmetric. This solution, by Theorem 2, also solves for equilibrium of the noisy-ranking model (1) when contestants' productions are identical.

[17] This special case of single-prize contests is well studied by Skaperdas (1996) and Clark and Riis (1996a, 1997) among others.

$$
\begin{aligned}
\Pr(\zeta_i \leq \zeta) &= \Pr(F_0^{-1}\left(e^{-\frac{1}{y_i}}\right) \leq \zeta) \\
&= \Pr(y_i \leq -1/\log F_0(\zeta)) \\
&= e^{-\frac{g_i(x_i)}{-1/\log F_0(\zeta)}} = e^{g_i(x_i)\log F_0(\zeta)} \\
&= [F_0(\zeta)]^{g_i(x_i)}.
\end{aligned} \tag{8}
$$

Equation 8 reveals the nature of ζ_i. Let $F_0(\cdot)$ be the distribution function of random variables v_i, $\forall i$. The random variable ζ_i is identical to the highest-order statistic of v_i s out of $g_i(x_i)$ independent draws.

When contestants are ranked in descending order by ζ_is, the following contest results. Each contestant i competes against the others by increasing the value of his/her output. Each contestant uses an x_i amount of effort to have $g_i(x_i)$ independent attempts. The function $g_i(x_i)$ is an increasing function of x_i. Each attempt allows the contestant to produce an output with a random value v_i, whose distribution follows c.d.f. $F_0(\cdot)$. The contest takes the contestant i's best shot, i.e., the highest realized value of v_is, as his/her entry in the contest. Contestants are ranked by ζ_is in descending order. The entry of contestant i, ζ_i, is simply distributed according to the c.d.f. $\Pr(\zeta_i \leq \zeta) = [F_0(\zeta)]^{g_i(x_i)}$. One should note that the interpretation of $g_i(x_i)$ as the number of draws requires that $g_i(x_i)$ maps from R^+ to the set of natural numbers. However, one may also slightly abuse this interpretation by allowing $g_i(x_i)$ to be a nonnegative real number.[18]

This contest only honors the best performance of each contestant, and ranks all contestants by their own "best shots." This fact allows us to name it a *FP Ranking Model* or a *"Best-Shot" Contest*. We call the c.d.f. $F_0(\cdot)$, the distribution of the underlying performance (of each single attempt). We obtain the following equivalence result:

Theorem 3 *The benchmark noisy-ranking model* (1) *is stochastically equivalent to the Favorable Performance Ranking Model with an arbitrary underlying performance distribution* $F_0(\cdot)$.

The (unique) statistical equivalence indicates that our benchmark model (1) and the popularly-adopted lottery contest models are underpinned by a "*favorable performance ranking*" (FP Ranking) model. Lottery contest models can be viewed to abstract competitive events that honor each contestant's best shot only.

"Best-shot" contests are not uncommon in reality. They can be created by a man-made rule. For instance, weightlifters are ranked in the Olympic Games by their most successful tries. More plausibly, this winning rule captures a natural regularity that is common in many real-world competitive events: on many occasions, only the best performance of a contestant can be observed. To provide an analogy of this argument, an architect would submit only his/her most successful design to a design competition.

[18] The integer problem of output $g_i(x_i)$ can also be interpreted from another angle. Suppose that the outputs $g_i(x_i)$ are practically distinguishable up to the degree of 10^{-N} where $N(\geq 1)$ is an integer. Our noisy-ranking model (2) is equivalent to $[y_i \cdot 10^N] = \breve{g}_i(x_i)\tilde{\varepsilon}_i = [g_i(x_i) \cdot 10^N]\tilde{\varepsilon}_i$, $\forall i \in \mathbf{I}$.For our ranking model, ranking y_i is equivalent to ranking $y_i \cdot 10^N$. Since $\breve{g}_i(x_i) = g_i(x_i) \cdot 10^N$ are integers, they can be interpreted as numbers of draws. Thus, the winning probability in a winner-take-all contest would be $p(i|\mathbf{x}) = \frac{\breve{g}_i(x_i)}{\sum_{j\in\mathbf{I}} \breve{g}_j(x_j)} = \frac{g_i(x_i)}{\sum_{j\in\mathbf{I}} g_j(x_j)}$, $\forall i \in \mathbf{I}$.

A lawyer will offer only the most favorable evidence in court, while the strongest case prevails.[19]

The research tournament model studied by Baye and Hoppe (2003) provides the most intuitive example of FP Ranking Models or "Best-Shot" Contests.[20] Each firm i hires n_i scientists (where n_i is an integer) to conduct R&D activities. Each scientist can come up with an idea, with the value of each idea following a continuous distribution with the common c.d.f. $F_0(\cdot)$. The firm picks the most valuable idea developed by its scientists as its bid to compete with other firms. The value of the best idea of a firm i is denoted by ζ_i. It follows a continuous distribution with the c.d.f. $G_i(\zeta) = \Pr(\zeta_i \leq \zeta) = [F_0(\zeta)]^{n_i}$. As shown by Baye and Hoppe (2003), as well as being implied by our previous analysis, a firm i 's probability of winning the competition are $n_i / \sum_{j=1}^{I} n_j$, which replicates the success function in a winner-take-all Tullock contest. Our analysis reveals the statistical and economic foundations that underpin this equivalence: this research tournament is exactly a contest that implements a FP ranking rule. It is this ranking rule that unifies these seemingly disparate contest/tournament models.

3 Discussion: "Dual" problem and antithesis

The equivalence between our ranking model and lottery contest models sheds light on the hidden mechanism in the black box of lottery contests. In this section, we further illustrate the economic implications of our noisy-ranking model. Our discussion proceeds dialectically. We first demonstrate the role of FP Ranking and the "Best-Shot" evaluation mechanism in the setting of a generalized race model (Dasgupta and Stiglitz 1980). We then show that this framework is stochastically "dual" to our benchmark contest model. Further, we elaborate on our argument by presenting its "antithesis" : a "weakest-link" contest (Hirshlefer and Riley 1992) is provided which cannot be abstracted as a lottery contest, as the FP Ranking is missing.

3.1 The "Dual" problem: a generalized race model

A race model is an abstraction of a competitive event where participants receive greater rewards when they accomplish a specific task sooner than others. One notable example is an R&D race where firms compete by developing an innovative technology, and the first innovator is rewarded by a patent. Dasgupta and Stiglitz (1980) first propose a winner-take-all patent race model to study drastic innovation competition. Baye and Hoppe (2003) show the equivalence between this framework and single-prize Tullock contest. In what follows, we demonstrate that this equivalence can be extended to more

[19] This court judging rule is extracted from Baye et al. (2005).

[20] Fullerton and McAfee (1999) proposed a very similar version of this research tournament model. They allowed the value of a firm's innovation x_i to follow the distribution $F^{z_i}(x_i)$, where the real number z_i is the amount of research conducted by the firm. By contrast, Baye and Hoppe (2003) explicitly assume the power term as an integer, and interpret it as the number of scientist. We follow Baye and Hoppe's (2003) approach as it is a more precise depiction of a "Best-Shot Contest."

general settings, which allow for multiple prizes. The equivalence is again underpinned by FP ranking rule.

We borrow the framework of Dasgupta and Stiglitz (1980) and Baye and Hoppe (2003). Our analysis, nevertheless, considers a more complete ranking among contestants, and therefore allows for multiple prizes. A multi-prize race model becomes appealing when analyzing the ramifications of a patent policy that rewards duplicators.[21] One may also imagine that a number of firms pursue a process R&D project. All the successful innovators may benefit from superior productivity. However, the earlier a firm succeeds in developing a cost-reducing technical solution, the higher is its profit.

Each of I contestants chooses a level of effort x_i to accomplish a task (e.g., an innovation). Contestant i accomplishes a task by time t_i with a probability (i.e., a Weibull minimum distribution) of

$$\Psi(t_i|x_i) = 1 - e^{-z_i(x_i)t_i}, x_i, t_i \geq 0, \tag{9}$$

where $z_i(x_i)$ is the hazard rate of contestant i, i.e., the conditional probability of accomplishing this task between t_i and $t_i + \Delta t_i$. Conditional on effort entry $\mathbf{x}$, t_is are independently distributed. The hazard rate $z_i(x_i)$ is a strictly increasing function of the expenditure x_i. Define $\mathbf{z}(\cdot) \triangleq (z_i(\cdot))$.

I prizes (denoted by $\mathbf{V} = (V_1, V_2, \ldots, V_I)$) are to be awarded to contestants.[22] Given effort entry $\mathbf{x} = (x_i)$, each conditional realization of (t_i) determines the ranks of the contestants and the prize distribution outcome. Denote this multi-prize race model by $R(\mathbf{I}, \mathbf{z}(\cdot), \mathbf{V})$ and a multiple-winner nested Tullock contest with contestants $\mathbf{I}$, technology $\mathbf{z}(\cdot)$ and prizes $\mathbf{V}$ by $C(\mathbf{I}, \mathbf{z}(\cdot), \mathbf{V})$.

Theorem 4 *A generalized race model $R(\mathbf{I}, \mathbf{z}(\cdot), \mathbf{V})$ is stochastically equivalent to a descending-order noisy-ranking contest (1) with the set of output functions $\mathbf{z}(\cdot)$ and i.i.d. noises ε that follow an extreme value type I (maximum) distribution. Hence, it is also equivalent to a generalized lottery contest model $C(\mathbf{I}, \mathbf{z}(\cdot), \mathbf{V})$.*

Theorem 4 states the stochastic equivalence of our multi-prize race model and a lottery contest model, as well as our benchmark ranking model. A dedicated technical proof is omitted for brevity.[23] However, the hidden ties that connect all these models surface as the discussion proceeds.

A race is essentially a noisy-ranking contest: contestants are ranked in ascending order based on the amounts of time they spend on the given task, and a contestant

[21] The winner-take-all rule in the patent system has been challenged by critics, and a high-profile policy debate has taken place in the U.S. and Europe over whether duplicators should be allowed to share the market with initial innovators (See Denicolò and Franzoni 2010).

[22] Prizes are allowed to carry zero value.

[23] The proof is available upon request.

receives a higher reward for using less time t_i. Recall that t_i is distributed by the Weibull (minimum) c.d.f. $\Pr(t \leq t_i) = 1 - e^{-z_i(x_i)t_i}$. We then consider a uniform increasing transformation of variables $t_i : \vartheta_i = F_0^{-1}(1 - e^{-t_i}), i = 1, 2, \ldots, I$, where $F_0(\cdot) : \mathbb{R}+ \to [0, 1]$ is an arbitrary continuous and strictly increasing function. Ranking these contestants by t_is is equivalent to ranking them by ϑ_is. A technique similar to that in Sect. 2.3 allows us to obtain the cumulative distribution function of ϑ_i:

$$\begin{aligned} \Pr(\vartheta_i \leq \vartheta) &= \Pr(F_0^{-1}(1 - e^{-t_i}) \leq \vartheta) \\ &= 1 - [1 - F_0(\vartheta)]^{z_i(x_i)}. \end{aligned} \tag{10}$$

$F_0(\cdot)$ can be interpreted as the distribution function of a random variable w_i. By Eq. 10, ϑ_i is identical to the lowest order statistics of w_is out of $z_i(x_i)$ independent draws. Contestants are ranked in ascending order by the realized minimum (ϑ_i). Hence, model (10) and the equivalent race model (9) are statistically "dual" to model (8) in Sect. 2.3, as well as the benchmark noisy-ranking model (1). Both (8) and (1) rank contestants in descending order by the realized maximum.

A race model naturally encapsulates a FP ranking mechanism. Ranking contestants in ascending order of ϑ_i (which is equivalent to ranking t_i) is, in terms of its statistical nature, a convenient depiction of the following contest. A contestant i expends a certain level of effort x_i to conduct a number $z(x_i)$ of parallel scientific experiments, where each experiment may lead to success at some point in time. The actual time that contestant i spends on the task is determined by his/her "best shot" or the "most FP": his/her first successful (parallel) experiment.

3.2 The "Antithesis" : an example of non-lottery contests

The FP Ranking model permits us to connect a variety of seemingly disparate models, but also imposes a limit on this relationship: The lottery contest framework does not include competitive events that do not honor "the most favorable shocks" when picking the winners.

We now present a contest model that has a different performance evaluation rule. It is adapted from the noisy-ranking contest model suggested by Hirshlefer and Riley (1992). Two contestants simultaneously submit their effort entries x_1 and x_2, and they are ranked by their composite outputs $y_i = q_i g_i(x_i)$. q_i is a random variable that follows a Weibull (minimum) distribution with c.d.f. $F(q_i) = 1 - e^{-aq_i}$; while $g_i(x_i)$ is a strictly increasing function of one's effort outlay x_i. The contestant with the higher y_i wins, so outputs are ranked in descending order. It can be easily verified that, given the set of effort entries, the contestants' ex ante winning odds are given by standard ratio-form success functions $\frac{g_i(x_i)}{g_1(x_1)+g_2(x_2)}, i = 1, 2$.

The equivalence of this model to a lottery contest does not hold when there are more than two contestants. When $I = 3$, and when only one prize is available, contestant 1 wins with a probability of

$$P_1 = 1 - \frac{g_2(x_2)}{g_1(x_1) + g_2(x_2)} - \frac{g_3(x_3)}{g_1(x_1) + g_3(x_3)} + \frac{g_2(x_2)g_3(x_3)}{g_1(x_1)g_3(x_3) + g_2(x_2)g_3(x_3) + g_1(x_1)g_2(x_2)}. \tag{11}$$

The proof is provided in Appendix A.3.

This setting results in a well-defined contest success function, but also departs from lottery contests. We reveal the source of dichotomy by comparing it to the race model. We know that in a race, contestants are ranked in ascending order by t_is, which can be alternatively rewritten as

$$t_i = q_i' h_i(x_i), \quad \forall i \in \mathbf{I}, \tag{12}$$

where $q_i' \in (0, \infty)$ is a random variable and $h_i(x_i) \triangleq z_i^{-1}(x_i)$.[24] By simple statistical facts, q_i' follows a Weibull (minimum) distribution with c.d.f. $1 - e^{-q_i}$.

Comparing t_i with y_i, and q_i' with q_i, respectively, readers will immediately realize that the model of Hirshlefer and Riley (1992) is no different from our race model except for the ranking rule. One wins in a race by achieving a smaller t_i (when ranked in ascending order). In contrast, a contestant in Hirshlefer and Riley (1992) wins by a larger y_i (when ranked in descending order). The Weibull (minimum) distribution indicates the distribution of the incidence of the "minimum" across a series of attempts. With ascending order ranking (e.g., in a race), the realized minimum arises out of the most favorable shock. In Hirshlefer and Riley (1992), the realized minimum, however, represents a contestant's "least FP" or his/her "worst shot" under the descending order ranking rule. The dichotomy in the underlying performance evaluation mechanisms drives this observed disparity.

Hence, the model of Hirshlefer and Riley (1992) is underpinned by a "worst-performance" ranking. This mechanism represents different economic activities from those that underlie lottery contests. Hirshlefer and Riley (1992) model depicts situations where the worst (instead of the best) performance of each contestant across multiple attempts counts most significantly when they are being ranked for winner selection. This winner selection rule embodies the widely quoted "wooden barrel principle" : the shortest plank determines the amount of water held in a wooden barrel.

4 Concluding remarks

This article sets forth a multi-prize contest model that links its prize distribution outcome to the ranking of contestants based on their noisy performance. The performance of a contestant is modeled as the sum of a deterministic output of his/her effort and a random component that follows a type I extreme value (maximum) distribution. Contestants exert their one-shot effort simultaneously, and contestants are rewarded by their ranks.

[24] Obviously, the function $h_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ strictly decreases with one's effort.

First, our noisy-ranking model delivers exactly the same success functions as a multi-prize lottery contest. This statistical equivalence provides an alternative interpretation of the multiple-winner nested contest model (Clark and Riis 1996b, 1998a): a (simultaneous) winner-selection mechanism (noisy-ranking rule) underlies its literally sequential lottery process. Second, we illuminate a hidden common thread that connects a wide variety of seemingly disparate contests (such as rent-seeking contests, patent races, and research tournaments). Underlying all these contests is a common noisy-ranking rule that honors the contestants' most FP. This article therefore provides a statistical micro foundation that underpins the family of commonly adopted lottery contest models.

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Appendix A.1: Proof of Lemma 2

The result of Lemma 2 relies on an important statistical property (**Property A1** below) of Type-I extreme value (maximum) distribution. Consider an arbitrary ranking outcome. Suppose that an arbitrary set of $K(1 \leq K \leq I-2)$ contestants are ranked from top 1 to top K by the amount of y_i. Let i_k indicate the index of the k-th ranked contestant. Define $\mathbf{I}_K = \{i_k, k = 1, \ldots, K\}$, which is the index set of the top ranked K contestants. We thus have $y_{i_1} \geq y_{i_2} \geq \cdots \geq y_{i_K} \geq y_j, \forall j \in \Omega_{K+1} \triangleq \mathbf{I} \backslash \mathbf{I}_K$. We now calculate the conditional probability of a contestant $n \in \Omega_{K+1}$ being ranked at $(K+1)$-th place. This probability is denoted by $p(n|\mathbf{N}_K, \mathbf{x}, Y_K)$, where $\mathbf{N}_K = (i_1, \ldots, i_K)$ denotes the sequence of the top K-ranked contestants, and $Y_K = (y_{i_1}, \ldots, y_{i_K})$ denotes the sequence of the observed outputs of the top K-ranked contestants.

We now present the following **Property A1**, which is the key step to establish Lemma 2.

Property A1 *For any given effort entries $\mathbf{x} \geq 0$ such that $\sum_{j \in \mathbf{N}} g_j(x_j) > 0$, the probability that a contestant $n \in \Omega_{K+1}$ is the $(K+1)$-th ranked, conditional on that contestants $i_1, i_2, \ldots, i_K$ are respectively ranked from top 1 to top K, can be expressed as*

$$p(n|\mathbf{N}_K, \mathbf{x}) = \frac{g_n(x_n)}{\sum_{j \in \Omega_{K+1}} g_j(x_j)}, \quad \forall n \in \Omega_{K+1}. \tag{13}$$

The following is the proof of **Property A1**. Since ε_i are i.i.d., the conditional cumulative distribution function of $\varepsilon_j, \forall j \in \Omega_{K+1}$ is described by

$$F(\varepsilon_j|\mathbf{N}_K, \mathbf{x}, Y_K) = F(\varepsilon_j|y_j < y_{i_K}) \\ = e^{-e^{-\varepsilon_j}}/e^{-e^{-\bar{\varepsilon}_j}}, \quad \varepsilon_j \in (-\infty, \bar{\varepsilon}_j), \forall j \in \Omega_{K+1}, \tag{14}$$

where $\bar{\varepsilon}_j \equiv \log y_{i_K} - \log g_j(x_j), \forall j \in \Omega_{K+1}$. It therefore yields the density function:

$$f(\varepsilon_j|\mathbf{N}_K, \mathbf{x}, Y_K) = e^{-\varepsilon_j - e^{-\varepsilon_j}}/e^{-e^{-\bar{\varepsilon}_j}}, \quad \varepsilon_j \in (-\infty, \bar{\varepsilon}_j), \forall j \in \Omega_{K+1}. \tag{15}$$

As implied by (14) and (15), the conditional distribution of $\varepsilon_j, \forall j \in \Omega_{K+1}$, only depends on the minimum of $\{y_{i_k}, k = 1, \ldots, K\}$, i.e., y_{i_K}, because y_i are ranked in descending order.

We first calculate $p(n|\mathbf{N}_K, \mathbf{x}, Y_K)$, which denotes the probability that a contestant $n \in \Omega_{K+1}$ is the $(K+1)$-th ranked conditioning on that contestants $\mathbf{N}_K = (i_1, i_2, \ldots, i_K)$ are, respectively, ranked from top 1 to top K and their observed outputs are Y_K. Note that $\varepsilon_n + \log g_n(x_n) - \log g_j(x_j) \leq \bar{\varepsilon}_j, \forall \varepsilon_n \in (-\infty, \bar{\varepsilon}_n), \forall j, n \in \Omega_{K+1}, j \neq n$. We thus have

$$\begin{aligned} &p(n|\mathbf{N}_K, Y_K, \mathbf{x}) \\ &= \Pr(\varepsilon_j \leq \varepsilon_n + \log g_n(x_n) - \log g_j(x_j), \forall j \in \Omega_{K+1}, j \neq n.) \\ &= \int_{-\infty}^{\bar{\varepsilon}_n} \left[\Pi_{j\in\Omega_{K+1}, j\neq n} F(\varepsilon_n + \log g_n(x_n) - \log g_j(x_j)|\mathbf{N}_K, \mathbf{x}, Y_K)\right] \\ &\quad \times f(\varepsilon_n|\mathbf{N}_K, \mathbf{x}, Y_K) d\varepsilon_n \\ &= \int_{-\infty}^{\bar{\varepsilon}_n} \left[\Pi_{j\in\Omega_{K+1}, j\neq n} e^{-e^{-(\varepsilon_n + \log g_n(x_n) - \log g_j(x_j))}}/e^{-e^{-\bar{\varepsilon}_j}}\right] e^{-\varepsilon_n - e^{-\varepsilon_n}}/e^{-e^{-\bar{\varepsilon}_n}} d\varepsilon_n \\ &= \left(\Pi_{j\in\Omega_{K+1}} 1/e^{-e^{-\bar{\varepsilon}_j}}\right) \int_{-\infty}^{\bar{\varepsilon}_n} \left[\Pi_{j\in\Omega_{K+1}, j\neq n} e^{-e^{-(\varepsilon_n + \log g_n(x_n) - \log g_j(x_j))}}\right] \\ &\quad \times e^{-\varepsilon_n - e^{-\varepsilon_n}} d\varepsilon_n \\ &= \left(\Pi_{j\in\Omega_{K+1}} 1/e^{-e^{-\bar{\varepsilon}_j}}\right) \int_{-\infty}^{\bar{\varepsilon}_n} \exp\left[-\varepsilon_n - e^{-\varepsilon_n} \cdot \left(1 + \sum_{j\in\Omega_{K+1}, j\neq n} \frac{g_j(x_j)}{g_n(x_n)}\right)\right] d\varepsilon_n. \end{aligned} \tag{16}$$

Let $\lambda_{n,\Omega_{K+1}} = \log\left(1 + \sum_{j\in\Omega_{K+1}, j\neq n} \frac{g_j(x_j)}{g_n(x_n)}\right) = \log\left(\sum_{j\in\Omega_{K+1}} \frac{g_j(x_j)}{g_n(x_n)}\right)$, then

$p(n|\mathbf{N}_K, Y_K, \mathbf{x})$

$$= \left(\Pi_{j\in\Omega_{K+1}} 1/e^{-e^{-\bar{\varepsilon}_j}}\right) \int_{-\infty}^{\bar{\varepsilon}_n} \exp\left[-\varepsilon_n - e^{-(\varepsilon_n-\lambda_{n,\Omega_{K+1}})}\right] d\varepsilon_n$$

$$= \left(\Pi_{j\in\Omega_{K+1}} 1/e^{-e^{-\bar{\varepsilon}_j}}\right) \exp(-\lambda_{n,\Omega_{K+1}}) \int_{-\infty}^{\bar{\varepsilon}_n-\lambda_{n,K}} \exp[-\varepsilon'_n - e^{-\varepsilon'_n}] d\varepsilon'_n$$

$$= \left(\Pi_{j\in\Omega_{K+1}} 1/e^{-e^{-\bar{\varepsilon}_j}}\right) \exp(-\lambda_{n,\Omega_{K+1}}) \exp[-e^{-(\bar{\varepsilon}_n-\lambda_{n,K})}]$$

$$= \left[g_n(x_n) \Big/ \sum_{j\in\Omega_{K+1}} g_j(x_j) \right] \cdot \left\{ \left(\Pi_{j\in\Omega_{K+1}} \exp[e^{-\bar{\varepsilon}_j}]\right) \exp[-e^{-(\bar{\varepsilon}_n-\lambda_{n,\Omega_{K+1}})}] \right\}$$

$$= \left[g_n(x_n) \Big/ \sum_{j\in\Omega_{K+1}} g_j(x_j) \right] \cdot \exp\left\{ \left(\sum_{j\in\Omega_{K+1}} e^{-\bar{\varepsilon}_j} \right) - e^{-(\bar{\varepsilon}_n-\lambda_{n,\Omega_{K+1}})} \right\}. \quad (17)$$

Note $\left(\sum_{j\in\Omega_{K+1}} e^{-\bar{\varepsilon}_j}\right) - e^{-\left(\bar{\varepsilon}_n-\lambda_{n,\Omega_{K+1}}\right)} = \left(\sum_{j\in\Omega_{K+1}} e^{-(\varepsilon_{n_K}+\log g_{n_K}(x_{n_K})-\log g_j(x_j))}\right)$ $-\exp\left\{-\left[\varepsilon_{n_K} + \log g_{n_K}(x_{n_K}) - \log g_n(x_n) - \left(\log\left(\sum_{j\in\Omega_{K+1}} g_j(x_j)\right) - \log g_n(x_n)\right)\right]\right\}$. It boils down to $\frac{e^{-\varepsilon_{n_K}}}{g_{n_K}(x_{n_K})}\left\{\sum_{j\in\Omega_{K+1}} g_j(x_j) - \sum_{j\in\Omega_{K+1}} g_j(x_j)\right\} = 0$.

(16) and (17) give

$$p(n|\mathbf{N}_K, Y_K, \mathbf{x}) = g_n(x_n) \Big/ \sum_{j\in\Omega_{K+1}} g_j(x_j). \quad (18)$$

(18) is a very strong result as it states that $p(n|\mathbf{N}_K, Y_K, \mathbf{x})$ does not depend on Y_K. Aggregating over all possible Y_K, we must have

$$p(n|\mathbf{N}_K, \mathbf{x}) = g_n(x_n) \Big/ \sum_{j\in\Omega_{K+1}} g_j(x_j), n \in \Omega_{K+1}, K = 1, \ldots, I-2,$$

which completes the proof of Property A1.

Property A1 provides the conditional probability of each contestant is being ranked $(K+1)$-th given that he/she is not among the first k highest. By Property A1, the conditional probability of a contestant being ranked as the $(K+1)$-th is independent of $(x_{i_1}, \ldots, x_{i_K})$, the effort entries of these top K-ranked contestants. Combining Lemma 1 and Property A1, Lemma 2 can be concluded immediately, which provides the probability distribution of every arbitrary complete ranking.

Appendix A.2: Proof of Theorem 2

Sufficiency: Consider model $\log y_i = \{\log g(x_i) - b\} + [\varepsilon_i + b] = \log\{g_i(x_i)/\exp(b)\} + (\varepsilon_i + b)$, which is equivalent to our original model $\log y_i = \log g(x_i) + \varepsilon_i$.

Note that $\varepsilon_i + b$ follows a standard type I extreme-value (maximum) distribution with with c.d.f. of $F(w) = e^{-e^{-w}}$, thus Theorem 1 holds for $\tilde{g}_i(x_i) = g_i(x_i)/\exp(b)$ for any pair (I, L) such that $I \geq 2$ and $I \geq L \geq 1$, which leads to the same prize allocation probabilities of Theorem 1. Thus, when $I \geq 3$ and $I \geq L \geq 1$, Theorem 1 means that if ε_i follows a type I extreme-value (maximum) distribution with c.d.f. of $F(\varepsilon_i) = e^{-e^{-(\varepsilon_i+b)}}$, our noisy-ranking model with observed performance y_i described by (1) is equivalent to the generalized multiple-winner nested contest model $C(\mathbf{I}, \mathbf{g}(\cdot), \mathbf{V})$. In other words, as long as ε_i follows a type I extreme-value (maximum) distribution with c.d.f. of $F(\varepsilon_i) = e^{-e^{-(\varepsilon_i+b)}}$, the two models deliver the same probability for every prize allocation.

Necessity: We first consider the case of $I \geq 3$ and $L = 1$, i.e., a single-prize contest with at least three contestants. Yellott (1977) has shown some nice results, which imply that when $I \geq 3$ and $L = 1$, the winning probabilities of every contestant i takes the form of $\frac{g_i(x_i)}{\sum_{j\in\mathbf{I}} g_j(x_j)}$ only if ε_is follow a type I extreme-value (maximum) distribution. According to his Definition 3 (p.120), our noisy-ranking model falls into the family of *Thurstone models*. His Lemma 1 (p.116) shows that the *Choice Axiom* (Axiom 1 in Luce 1959) is satisfied if and only if there exists a set of scale values $(v_1, v_2, \ldots, v_n)$ such that contestant i is selected as the winner with probability $\frac{v_i}{\sum_{j\in\mathbf{I}} v_j}$. His Theorem 5 (p.135) shows that a Thurstone model satisfies the *Choice Axiom* if and only if the random noises ε_is follow a type I extreme-value (maximum) distribution, i.e., the cumulative distribution function of ε_i is $F(\varepsilon_i) = e^{-e^{-(a\varepsilon_i+b)}}$ with $a > 0$, $\varepsilon_i \in (-\infty, +\infty)$, $\forall i \in \mathbf{I}$. Consider model $\log[(y_i)^a] = \{\log[g_i(x_i)]^a - b\} + [a\varepsilon_i + b] = \log\{[g_i(x_i)]^a/\exp(b)\} + [a\varepsilon_i + b]$,which is equivalent to our original model $\log y_i = \log g(x_i) + \varepsilon_i$. Clearly, as $a > 0$, for our ranking model, ranking the contestants according y_i is equivalent to ranking them according to $(y_i)^a$. Note that $a\varepsilon_i + b$ follows a standard type I extreme-value (maximum) distribution with with c.d.f. of $F(w) = e^{-e^{-w}}$. The same procedure of showing Lemma 1 would lead to that contestant i's winning probability is given by $\frac{(g_i(x_i))^a/\exp(b)}{\sum_{j\in\mathbf{I}}(g_j(x_j))^a/\exp(b)} = \frac{(g_i(x_i))^a}{\sum_{j\in\mathbf{I}}(g_j(x_j))^a}$. For this probability to coincide with $\frac{g_i(x_i)}{\sum_{j\in\mathbf{I}} g_j(x_j)}$ for a general set of production functions $\mathbf{g}$ and any set of effort profile $\mathbf{x}$, we must have $a = 1$. Therefore, the uniqueness of type I extreme-value (maximum) distribution for ε_i with c.d.f. of $F(\varepsilon_i) = e^{-e^{-(\varepsilon_i+b)}}$ is guaranteed, which leads to the ratio-form winning probability $\frac{g_i(x_i)}{\sum_{j\in\mathbf{I}} g_j(x_j)}$ for $L = 1$ for our noisy-ranking model. This result means that if the probability that contestant i's performance is the highest among all contestants takes a form of $\frac{g_i(x_i)}{\sum_{j\in\mathbf{I}} g_j(x_j)}$ for our noisy-ranking model, then ε_is must follow a type I extreme-value (maximum) distribution with c.d.f. of $F(\varepsilon_i) = e^{-e^{-(\varepsilon_i+b)}}$.

We now turn to the case where $I \geq 3$ and $I \geq L \geq 2$. Suppose the equivalence holds for $L \geq 2$. Given that the equivalence holds, we have that the probability of any prize distribution outcome $\{i_k\}_{k=1}^{L}$ is given by (7) for our noisy-ranking model. Consider $L' = L - 1$. The probability of any prize distribution outcome $\{i_k\}_{k=1}^{L'}$, i.e., contestants $\{i_k\}_{k=1}^{L'}$ win the first L' prizes, must equal to the probability that contestants $\{i_k\}_{k=1}^{L'}$ win the first L' prizes while the L-th winner can be any one among those who is not among $\{i_k\}_{k=1}^{L'}$. Thus, according to (7), we must have

$$\begin{aligned}
p(\{i_k\}_{k=1}^{L'}) &= \sum_{i_L \in \mathbf{I}\backslash\{i_k\}_{k=1}^{L'}} p(\{i_k\}_{k=1}^{L'}, i_L) \\
&= \sum_{i_L \in \mathbf{I}\backslash\{i_k\}_{k=1}^{L'}} \left\{ \left[\Pi_{k=1}^{L'} \frac{g_{i_k}(x_{i_k})}{\sum_{k'=k}^{I} g_{i_{k'}}(x_{i_{k'}})} \right] \cdot \frac{g_{i_L}(x_{i_L})}{\sum_{k' \in \mathbf{I}\backslash\{i_k\}_{k=1}^{L'}} g_{i_{k'}}(x_{i_{k'}})} \right\} \\
&= \left[\Pi_{k=1}^{L'} \frac{g_{i_k}(x_{i_k})}{\sum_{k'=k}^{I} g_{i_{k'}}(x_{i_{k'}})} \right] \cdot \sum_{i_L \in \mathbf{I}\backslash\{i_k\}_{k=1}^{L'}} \frac{g_{i_L}(x_{i_L})}{\sum_{k' \in \mathbf{I}\backslash\{i_k\}_{k=1}^{L'}} g_{i_{k'}}(x_{i_{k'}})} \\
&= \Pi_{k=1}^{L'} \frac{g_{i_k}(x_{i_k})}{\sum_{k'=k}^{I} g_{i_{k'}}(x_{i_{k'}})}.
\end{aligned}$$

Continuing this process, we will have that the probability that contestant i wins the first prize V_1 is given by $\frac{g_i(x_i)}{\sum_{j\in\mathbf{I}} g_j(x_j)}$, $\forall i \in \mathbf{I}$ for our noisy-ranking model. Note that for our noisy-ranking model, the probability that contestant i wins the first prize is the probability that the contestant's performance y_i is the highest among all contestants. The discussion of the previous paragraph has revealed that if the probability that contestant i's performance is the highest among all contestants takes a form of $\frac{g_i(x_i)}{\sum_{j\in\mathbf{I}} g_j(x_j)}$ for our noisy-ranking model, then ε_is must follow a type I extreme-value (maximum) distribution. Therefore, the uniqueness of type I extreme-value (maximum) distribution for ε_is with c.d.f. of $F(\varepsilon_i) = e^{-e^{-(\varepsilon_i+b)}}$ must follow.

Appendix A.3: Proof of the "Antithesis"

We now prove that when there are three contestants ($I = 3$), the noisy-ranking contest model presented in Sect. 3.2 does not deliver a standard lottery contest. Adapted from Hirshlefer and Riley (1992), we utilize a formulation with multiplicative noise term:

$$y_i = q_i g_i(x_i), \tag{19}$$

where the q_i follows a Weibull minimum distribution with c.d.f. $1 - e^{-q_i}$. (19) can be equivalently expressed as

$$\log y_i = \log g_i(x_i) + \log q_i.$$

This distribution of $\varepsilon_i \triangleq \log q_i$ is a type I extreme-value (minimum) distribution. The c.d.f. and p.d.f. of ε_i are thus $F(\varepsilon_i) = 1 - \exp(-e^{\varepsilon_i})$, and $f(\varepsilon_i) = e^{\varepsilon_i - e^{\varepsilon_i}}$, respectively. Consider the case of three contestants ($I = 3$). Given the effort x_i, contestant 1 wins with the following probability:

$$\int_{-\infty}^{+\infty}[\Pi_{j=2,3}F(\varepsilon_1+\log g_1(x_1)-\log g_j(x_j))]f(\varepsilon_1)d\varepsilon_1$$

$$=\int_{-\infty}^{+\infty}[(1-\exp(-e^{\varepsilon_1+\log g_1(x_1)-\log g_2(x_2)}))(1-\exp(-e^{\varepsilon_1+\log g_1(x_1)-\log g_3(x_3)})]$$

$$\times e^{\varepsilon_1-e^{\varepsilon_1}}d\varepsilon_1$$

$$=1-\int_{-\infty}^{+\infty}[\exp(-e^{\varepsilon_1+\log g_1(x_1)-\log g_2(x_2)})-\exp(-e^{\varepsilon_1+\log g_1(x_1)-\log g_3(x_3)})]$$

$$\times e^{\varepsilon_1-e^{\varepsilon_1}}d\varepsilon_1$$

$$+\int_{-\infty}^{+\infty}\exp(-e^{\varepsilon_1+\log g_1(x_1)-\log g_2(x_2)})\cdot\exp(-e^{\varepsilon_1+\log g_1(x_1)-\log g_3(x_3)})$$

$$\cdot e^{\varepsilon_1-e^{\varepsilon_1}}d\varepsilon_1$$

$$=1-\int_{-\infty}^{+\infty}\exp\left(\varepsilon_1-e^{\varepsilon_1}\left(1+\frac{g_1(x_1)}{g_2(x_2)}\right)\right)d\varepsilon_1-\int_{-\infty}^{+\infty}\exp\left(\varepsilon_1-e^{\varepsilon_1}\left(1+\frac{g_1(x_1)}{g_3(x_3)}\right)\right)d\varepsilon_1$$

$$+\int_{-\infty}^{+\infty}\exp\left(\varepsilon_1-e^{\varepsilon_1}\left(1+\frac{g_1(x_1)}{g_2(x_2)}+\frac{g_1(x_1)}{g_3(x_3)}\right)\right)d\varepsilon_1$$

$$=1-\int_{-\infty}^{+\infty}\exp\left(\varepsilon_1-e^{\varepsilon_1+\log\left(1+\frac{g_1(x_1)}{g_2(x_2)}\right)}\right)d\varepsilon_1-\int_{-\infty}^{+\infty}\exp\left(\varepsilon_1-e^{\varepsilon_1+\log\left(1+\frac{g_1(x_1)}{g_3(x_3)}\right)}\right)d\varepsilon_1$$

$$+\int_{-\infty}^{+\infty}\exp\left(\varepsilon_1-e^{\varepsilon_1+\log\left(1+\frac{g_1(x_1)}{g_2(x_2)}+\frac{g_1(x_1)}{g_3(x_3)}\right)}\right)d\varepsilon_1$$

$$=1-\frac{g_2(x_2)}{g_1(x_1)+g_2(x_2)}-\frac{g_3(x_3)}{g_1(x_1)+g_3(x_3)}$$

$$+\frac{g_2(x_2)g_3(x_3)}{g_1(x_1)g_3(x_3)+g_2(x_2)g_3(x_3)+g_1(x_1)g_2(x_2)}.$$

References

Amegashie JA (2000) Some results on rent-seeking contests with shortlisting. Public Choice 105:245–253

Barut Y, Kovenock D (1998) The symmetric multiple-prize all-pay auction with complete information. Eur J Polit Econ 14:627–644

Baye M, Hoppe H (2003) The strategic equivalence of rent-seeking, innovation, and patent-race games. Games Econ Behav 44:217–226

Baye M, Kovenock D, de Vries C (2005) Comparative analysis of litigation systems: an auction theoretic approach. Econ J 115:583–601

Beggs S, Cardell S, Hausman J (1981) Assessing the potential demand for electric cars. J Econometrics 16:1–19
Clark JJ, Riis C (1996a) On the win probability in rent-seeking games, mimeo
Clark DJ, Riis C (1996b) A multi-winner nested rent-seeking contest. Public Choice 87:177–184
Clark DJ, Riis C (1997) Contest success functions: an extension. Econ Theory 11:201–204
Clark DJ, Riis C (1998a) Influence and the discretionary allocation of several prizes. Eur J Polit Econ 14:605–615
Clark DJ, Riis C (1998b) Competition over more than one prize. Am Econ Rev 88:276–289
Dasgupta P, Stiglitz J (1980) Industrial structure and the nature of innovative activity. Econ J 90:266–293
Denicolò V, Franzoni LA (2010) On the winner-take-all principle in innovation races. J Eur Econ Assoc 8(5):1133–1158
Fu Q, Lu J (2009a) The beauty of bigness: on optimal design of multi-winner contests. Games Econ Behav 66:146–161
Fu Q, Lu J (2009b) The optimal multiple-stage contest. Econ Theory (forthcoming)
Fullerton RL, McAfee RP (1999) Auctioning entry into tournaments. J Polit Econ 107:573–605
Glazer A, Hassin R (1988) Optimal contests. Econ Inq 26:133–143
Hann IH, Hui KL, Lee T, Png IP (2008) Consumer privacy and marketing avoidance: a static model. Manage Sci 54(6):1094–1103
Hirshlefer J, Riley JG (1992) The analysis of uncertainty and information. Cambridge University Press, Cambridge, UK
Jia H (2008) A stochastic derivation of the ratio form of contest success functions. Public Choice 135: 125–130
Konrad K (2009) Strategy and dynamics in contests. Oxford University Press, Oxford
Lazear EP, Rosen S (1981) Rank-order tournaments as optimum labor contracts. J Polit Econ 89:841–864
Luce RD (1959) Individual choice behavior. Wiley, New York
Luce RD, Suppes P (1965) Preference, utility, and subjective probability. In: Luce RD, Bush RR, Galanter E (eds) Handbook of mathematical psychology, vol III. John Wiley and Sons, New York
McFadden D (1973) Conditional Logit Analysis of Qualitative Choice Behavior. In: Zarembka P (ed) Frontier in econometrics. Academic, New York
McFadden D (1974) The measurement of urban travel demand. J Public Econ 3:303–328
Moldovanu B, Sela A (2001) The optimal allocation of prizes in contests. Am Econ Rev 91:542–558
Moldovanu B, Sela A, Shi X (2007) Contests for status. J Polit Econ 115:338–363
Morgan J, Várdy F (2007) The value of commitment in contests and tournaments when observation is costly. Games Econ Behav 60:326–338
Siegel R (2009) All-pay contests. Econometrica 77(1):71–92
Skaperdas S (1996) Contest success function. Econ Theory 7:283–290
Szymanski S, Valletti TM (2005) Incentive effects of second prizes. Eur J Polit Econ 21:467–481
Wärneryd K (2000) In defense of lawyers: moral hazard as an aid to cooperation. Games Econ Behav 33:145–158
Yates AJ, Heckelman JC (2001) Rent-setting in multiple winner rent-seeking contests. Eur J Polit Econ 17:835–852
Yellott JI (1977) The relationship between Luce's choice axiom, thurstone's theory of comparative judgment, and the double exponential distribution. J Math Psychol 15:109–177
Yildirim H (2005) Contests with multiple rounds. Games Econ Behav 51:213–227