Multi-Prize Contests with Risk-Averse Players*

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Abstract

This paper studies a multi-prize imperfectly discriminatory contest with symmetric risk-averse contestants. Adopting a multiple-winner nested Tullock contest model, we first establish the existence and uniqueness of a symmetric pure-strategy equilibrium under plausible conditions. We then investigate the optimal prize allocation in the contest. Our analysis provides a formal account of the incentive effects triggered by a variation in the prevailing prize structure when contestants are risk averse. We demonstrate that contestants' incentive subtly depends not only on the degree of a contestant's risk aversion (i.e., the second-order property of utility function) but also that of his prudence (i.e., the third-order property of utility function). The former affects the marginal benefit of effort, while the latter affects the marginal cost. We derive sufficient conditions under which a single-(multi-)prize contest would emerge in the optimum when the contest designer aims to maximize total effort. We also discuss in depth the roles played by risk aversion and prudence in optimal prize allocation.

Keywords: Risk Aversion; Bidding Equilibrium; Prize Allocation; Multiple Prizes; Contest Design.

JEL Classification Codes: C72, D72, D81, J31.

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1 Introduction

Many competitive activities resemble a contest. Interest groups lobby to influence policies; litigants engage in legal battles for favorable court rulings; firms invest in R&D to secure market leadership; workers climb a firm's hierarchical ladder for promotion to higher rungs; and students vie for seats at elite colleges. All of these scenarios exemplify contests: Economic agents strive to get ahead of their peers, and scarcely supplied prizes are available only to top performers.

Investment in such competition is both rewarding and risky: On the one hand, effort improves one's relative standing in the competition and thus increases the odds of securing a prize; on the other hand, input is nonrecoverable regardless of the outcome of the competition. The tantalizing but limited rewards lure economic agents to pursue superiority relentlessly; this, amid the uncertainty inherent in the competition, could discourage upfront investment to avoid loss. The tension caused by the gambling nature of a contest game particularly concerns risk-averse participants who are sensitive to variance in terminal payoffs, and governs their strategic choice of effort in response to different prize structures of the contest.

Imagine a shift of prize money across prizes of different ranks: This alters a risk-averse contestant's wealth distribution across states—i.e., the outcomes of being ranked in different places—and causes nonlinear variations in his utility (and marginal utility) evaluated in these states, which ultimately affects his effort incentive in the competition. Two fundamental economics inquiries naturally ensue. First, how would risk-averse contestants strategically respond to a variation in the prevailing prize structure of the contest, and how does their risk attitude affect the trade-off between benefit and cost in their equilibrium effort choice, compared to the risk-neutral counterpart? Second, how should a contest designer, with a fixed prize purse, fine-tune the prize structure of a contest to better incentivize effort supply? Should she concentrate her entire prize purse on the top prize, or split it to lower the bar for reward?

This paper sets out to address these questions. The majority of previous studies in the contest literature assume either/both risk-neutral contestants or/and a winner-take-all prize structure.[1] Our analysis demonstrates that risk aversion fundamentally alters the nature of contestants' trade-off in effort choice, which casts doubt into the conventional wisdom that a winner-take-all prize structure provides superior incentive. For this purpose, we set up a multi-prize contest model with risk-averse players and establish the existence and uniqueness of a pure-strategy bidding equilibrium under plausible conditions. This lays a foundation for a formal account of the incentive effect of prize structure under risk aversion, and further,

[1] Notable exceptions are discussed later in this section.

allows us to identify relevant conditions under which a single-(multi-)prize contest emerges as the optimum.

Prize structure has long been recognized as an important structural element of a contest that could be manipulated to boost performance. The literature has conventionally espoused the high-power incentive provided by a single grand prize. Rosen (1986), for instance, proposed the celebrated thesis that prize money should be concentrated on a top final prize awarded to the grand winner.[2] The famous Netflix Prize, which sought algorithms of higher predictive power, provides one salient example: It awarded a US$1M grand prize to the BellKor's Pragmatic Chaos team. Competitions that award several prizes, however, are also widespread in practice. The most intuitive examples are seen in sporting events—e.g., the Olympic Games—that typically award three (gold, silver, and bronze) medals; in addition, athletes earn professional ranking points based on the stages of the tournament they manage to reach, even if they fail to win one of these medals. The various competitions hosted by the XPRIZE Foundation typically split the prize purse among a few winners: In the recent Water Abundance XPRIZE, for instance, the California-based Skysource/Skywater Alliance secured a grand prize of $1.5M for developing an easily deployable high-volume water generator; a second team, Hawaii-based JMCC WING, was awarded a $150K prize for its ingenious technological approach.[3] In this paper, we explore the implications of players' risk attitude for prize structure.

Incentive Effect of Prize Structure under Risk Aversion We now briefly discuss how risk attitude reshapes contestants' response to a variation in prize structure. We adopt the popularly studied multiple-winner nested Tullock contest (Clark and Riis, 1996) to model a prize distribution mechanism that awards multiple prizes. Following Konrad and Schlesinger (1997), Treich (2010), and Cornes and Hartley (2012), we assume that a contestant, indexed by $i \in \{1, \dots, N\}$, has a smooth concave function $u(\cdot)$ and an expected utility

$$\sum_{m=1}^{N} \left[P_m^i \times u(w + V_m - e^i) \right],$$

where V_m is the prize for the mth rank, P_m^i the probability of his achieving the mth rank, e^i his effort entry, and w the initial endowment of wealth. Begin with a winner-take-all contest, and imagine a hypothetical shift of a small amount of prize money from the single top prize to a prize for the runner-up. As we illustrate below, the shift of prize money triggers a three-

[2]The winner-take-all principle is also affirmed in imperfectly discriminatory contest settings, e.g., Clark and Riis (1996, 1998), Fu and Lu (2012a), and Schweinzer and Segev (2012).

[3]Google Code Jam, a renowned annual programming competition of algorithmic challenges, also selects multiple prize winners.

way trade-off on a contestants' effort incentive, and the aggregate effect subtly depends not only on the degree of a contestant's risk aversion (i.e., the second-order property of utility function) but also that of his prudence (i.e., the third-order property of utility function).[4]

A negative effect is immediate: Setting a second prize tends to soften the competition, which allows a contestant to be rewarded without outperforming all others, thereby diminishing the marginal benefit of effort and weakening his incentive to leapfrog. The negative effect underpins the usual rationale of the winner-take-all principle (see, for example, Fu and Lu, 2012a, and Schweinzer and Segev, 2012). Two additional competing forces, however, may loom large when contestants are risk averse and could possibly counteract the negative incentive effect.

First, a risk-averse contestant, because of the concavity of his utility function, tends to discount the extra utility gain from a given wealth increase in more favorable states compared to his risk-neutral counterpart. With the shift of prize money, the contestant ends up with a smaller prize and, therefore, a utility loss in the event that he is the top performer, while he perceives extra utility in the state of being ranked in second place. The latter gain may more than offset the former loss when risk aversion is in place, which is impossible under risk neutrality. This effect, at least partly, offsets the above-mentioned negative effect. As a result, a given amount of prize money, when allocated to the second prize, may incentivize risk-averse contestants more effectively than being concentrated on the top prize. The ultimate effect on the marginal benefit of effort is ambiguous a priori and depends on the second-order property of the utility function.

Second, the hypothetical shift of prize money affects not only the marginal benefit of contestants' effort but also the marginal cost. Effort depletes contestants' wealth. Without risk aversion, one's marginal cost of effort is exogenously given and independent of the prevailing prize structure. With risk aversion, however, one's "marginal effort cost" boils down to the marginal disutility caused by wealth reduction aggregated in expectation over all possible states, i.e., at all possible ranks. The marginal effort cost curve is thus endogenously determined by the prize structure because the marginal disutility evaluated in each state depends on the associated prize. When the aforementioned hypothetical shift of prize money takes place, the contestant perceives an increase in his marginal disutility for the state of winning the top prize due to concavity, as well as a decrease for the state of obtaining the second rank. The aggregate effect thus depends on the second-order property of marginal utility, i.e., the third-order property of the utility function. When contestants are prudent, i.e., with a convex marginal utility, a contestant is more sensitive to downward risk, in which case the latter decrease more than offsets the former increase and reduces marginal effort

[4] An economic agent is called "prudent" when the marginal utility function $u'(\cdot)$ is convex. Prudence is interpreted as a measure of the "sensitivity of the optimal choice of a decision variable to risk" (see Kimball, 1990). It is well known that a higher degree of prudence gives rise to a precautionary saving motive.

cost. The second prize gives rise to a less polarized wealth distribution across states, which, analogous to precautionary saving, reduces the downward risk to the contestant and limits the overall disutility for effort. This would in turn encourage risky investment in the contest.

Snapshots of Results Assuming homogeneous contestants, our study verifies that a unique symmetric pure-strategy equilibrium exists under a broad range of contest technologies when contestants exhibit nonincreasing absolute risk aversion or have quadratic preferences. We do not impose specific restrictions on our setting to solve for the equilibrium in closed form. The equilibrium condition, however, suffices to shed light on the nature of contestants' effort incentive under risk aversion. The aforementioned three-way trade-off yields general insights for the implications of risk attitude for prize allocation.

Overall, we demonstrate that multiple prizes are more likely to emerge in the optimum when contestants are more risk averse and prudent. We formally derive sufficient conditions under which a single-/multi-prize contest is optimal. When contestants exhibit sufficiently mild risk aversion and relatively weak prudence, the two positive effects—which favor multiple prizes—are inadequate to counteract the negative effect. As a result, the optimal contest does not depart from the prediction obtained under risk neutrality, and again embraces the winner-take-all principle (Proposition 2). In contrast, when contestants are prudent and sufficiently risk averse, a multi-prize contest outperforms its single-prize counterpart (Proposition 3). In this case, higher-degree risk aversion strengthens the first positive effect—by which a second prize magnifies the marginal benefit of effort—while prudence catalyzes the second positive effect, in which case awarding a second prize reduces marginal cost. In particular, we demonstrate that positive prizes for lower ranks are likely in the optimum due to the cost-reducing effect of prudence, even if they provide direct negative incentive and reduce the marginal benefit of effort. The roles played by risk aversion and prudence are, respectively, discussed in detail in Sections 4.3.1 and 4.3.2.

We show that the optimum requires a strictly decreasing prize series, in that all positive prizes must be assigned in a descending order, with a larger prize awarded to a higher-ranked contestant (Proposition 1). The paper does not set out to provide a closed-form solution to the optimal prize schedule, which is available only in specific setup. However, we derive an upper bound for the number of prizes in the optimum (Proposition 4). A few comparative statics are provided. In particular, we demonstrate that a larger prize purse facilitates multiple prizes; a similar effect can be seen in an increase in the number of contestants (Corollary 2).

Contribution and Relation to Literature Our paper is related to three strands of the literature. First, it contributes to the literature that explores the fundamentals of equilibria in imperfectly discriminatory contest games. The existence and uniqueness of bidding equilibria have been thoroughly studied in winner-take-all Tullock contests with risk-neutral contestants.[5] An increasing number of studies introduce more general preferences into contest models.[6] Skaperdas and Gan (1995) identify the conditions under which pure-strategy equilibria exist in two-player contests with a general contest success function when contestants exhibit constant absolute risk aversion (CARA). Cornes and Hartley (2003) allow for multiple heterogeneous contestants with CARA utility and verify that a unique equilibrium exists in a lottery contest. Assuming a concave impact function, Cornes and Hartley (2012) show that risk aversion may lead to multiple equilibria in a general lottery contest with homogeneous contestants: Both symmetric and asymmetric equilibria may arise. Yamazaki (2009) verifies that a unique pure-strategy equilibrium exists in a general lottery contest when contestants have nonincreasing absolute risk aversion. Jindapon and Yang (2017) further extend this stream of research by allowing for non-cash prize, risk-loving contestants, and sequential bidding. Again, all of these studies assume a single prize. The framework of multiple-winner nested Tullock contests (Clark and Riis, 1996, 1998) has been popularly adopted to model contests that award several prizes. Somewhat surprisingly, the existence of equilibrium in the model was not formally established until the recent contribution of Fu, Wu and Zhu (2019). In this study, we further identify the condition under which a unique symmetric pure-strategy equilibrium exists for homogeneous contestants with nonincreasing absolute risk aversion or quadratic utility.

Second, our paper contributes to the literature that explores the strategic substance of contest games under risk aversion. Focusing on a symmetric contest with a general contest success function, Konrad and Schlesinger (1997) show that the impact of risk aversion on contestants' effort is ambiguous in a symmetric contest with general concave utility. In a similar setup, Treich (2010) concludes that risk aversion always leads to less effort when contestants are prudent.[7] Sahm (2017) shows in a general lottery contest that contestants are disadvantaged when they exhibit stronger aversion to downward risk, i.e., a higher degree of prudence. Schroyen and Treich (2016) explore how wealth endowment affects risk-averse contestants' effort incentives in a two-player contest. To the best of our knowledge, our paper

[5] Szidarovszky and Okuguchi (1997) and Cornes and Hartley (2005), for instance, establish the existence and uniqueness of interior equilibrium for a sufficiently noisy contest, i.e., the discriminatory power parameter $r \leq 1$. Baye, Kovenock and de Vries (1994), Alcalde and Dahm (2010), Ewerhart (2015, 2017a,b), and Feng and Lu (2017) venture into contests with large but finite discriminatory power, i.e., the discriminatory power parameter $r > 1$.

[6] Research on risk aversion in contests dates back to Hillman and Katz (1984).

[7] In a model similar to Konrad and Schlesinger (1997) and Treich (2010), Liu, Meyer, Rettenmaier and Saving (2018) consider a scenario in which only the winner of a contest pays for the resources used to compete. They show that when payment is contingent on winning, the effect of risk aversion is in the opposite direction of what occurs when costs are paid upfront regardless of the outcome.

is the first to explore the impact of prize structure in Tullock contests with risk aversion.[8]

Third, our paper adds to the literature on optimal prize allocation in contests.[9] In a multi-winner nested Tullock contest model, Clark and Riis (1998) show that a winner-take-all contest is optimal when homogeneous contestants are risk neutral and the cost function is linear. In a similar setup, Fu and Lu (2012a) consider a multi-stage sequential-elimination contest and establish a hierarchical winner-take-all principle, which requires that only a single grand prize be awarded in the finale of the contest when the winner-selection mechanism is sufficiently noisy.[10] Schweinzer and Segev (2012) further extend Clark and Riis (1998) by allowing for nonlinear cost function and reaffirm that the contest designer would optimally award the entire prize money to the top performer, provided that a symmetric pure-strategy equilibrium exists.[11,12]

A few studies examine the optimal prize allocation in the setting of all-pay auctions. Glazer and Hassin (1988) pioneered in this stream of research. In an incomplete-information all-pay auction model, Moldovanu and Sela (2001) demonstrate that a winner-take-all contest can be suboptimal when effort costs are convex. Fang, Noe and Strack (forthcoming) consider a complete-information setting and find a demoralizing effect of a more unequal prize structure for homogeneous contestants with convex costs.[13,14] In a large-contest framework (Olszewski and Siegel, 2016), Olszewski and Siegel (2018) show that numerous heterogeneous prizes can be optimal when contestants' valuations for the prize are concave and effort costs are convex.

Krishna and Morgan (1998), Akerlof and Holden (2012), and Drugov and Ryvkin (2018) explore optimal prize allocation in tournaments with additive noises. In particular, Krishna and Morgan establish a winner-take-all principle for optimal prize allocation in small tournaments. Akerlof and Holden highlight the contrasting roles played by rewards for top

[8]In contrast to this strand of literature that focuses on effort incentives, March and Sahm (2018) examine how risk aversion affects the selection efficiency of a contest in a two-player setting.

[9]See Sisak (2009) for a comprehensive survey of this topic.

[10]In contrast, Szymanski and Valletti (2005) allow for heterogeneous contestants, and in a three-player case, show that a second prize can incentivize contestants more effectively.

[11]Schweinzer and Segev (2012) show that a multiple-prize structure dampens incentives compared with a single prize, and thus can be used to sustain a pure-strategy equilibrium when such an equilibrium fails to exist under a winner-take-all prize structure.

[12]Liu and Treich (2019) study a multi-competition contest that allows a contestant to have multiple shots to win prizes, and compare it with a contest that allows all losers to receive an equal share of a prize. They show that these alternative schemes may outperform a winner-take-all contest under risk aversion and prudence.

[13]Moldovanu and Sela (2006) further allow the designer to decide on both the division of prize money and whether to embed a two-stage architecture in the contest, which eliminates a subset of contestants in a preliminary stage. They show that the optimum depends on the design objectives, number of contestants, and shape of the effort cost functions.

[14]Moldovanu, Sela and Shi (2007) study the optimal distribution of status categories in an environment in which contestants value status as an intangible prize. Moldovanu, Sela and Shi (2012), Thomas and Wang (2013), and Liu, Lu, Wang and Zhang (2018) further expand the design space by allowing for negative prizes.

performers vis-à-vis punishment for bottom performers. In contrast to the majority of this literature, the designer is not constrained by a fixed prize budget. Drugov and Ryvkin explore how the distribution of noise terms affects optimal prize allocation and demonstrate that the presence of heavy tails may compel the designer to split her prize purse into several uniform prizes. Drugov and Ryvkin (2019) further allow for risk aversion and characterize a sufficient condition under which winner-take-all principle can be retained.

Among these studies, Krishna and Morgan (1998), Glazer and Hassin (1988), Akerlof and Holden (2012), Drugov and Ryvkin (2019) allow for nonlinear valuation for prizes and hence include risk aversion. However, all these studies assume that the utility from prize is additively separated from effort cost. In this case, a variation in prize structure does not affect each contestant's marginal cost of effort, which nullifies the effect of prudence. Our results differ from those obtained under separable utility, but our analysis also sheds light on that setting. In particular, nonseparable utility unleashes the incentive effect of prudence, which leads to more even prize allocation profiles. We show that positive prizes for lower ranks are likely under nonseparable utility even if they provide direct negative incentive, which is impossible under separable utility. This nuance is discussed in more detail in Section 4.3.2.

This paper proceeds as follows. In Section 2, we set up the model. In Section 3, we characterize the unique symmetric pure-strategy equilibrium of the contest game. In Section 4, we lay out the main analysis and establish the conditions under which single (multiple) prize(s) would be optimal. We further discuss the role of contestants' risk aversion and prudence in determining the optimal prize schedule. In Section 5, we summarize our main findings and suggest directions for future research. All proofs are relegated to an appendix.

2 The Model

A contest involves $N \geq 3$ homogeneous contestants, indexed by $i \in \mathcal{N} \equiv \{1, \cdots, N\}$. Each is endowed with an initial income $w > 0$. A total of N prizes are to be given away in the contest based on contestants' ranks; they are ordered in a decreasing prize series $V_1 \geq \ldots \geq V_N \geq 0$, with strict inequality holding for at least one.[15] Contestants simultaneously commit to their costly effort e^is to vie for these prizes, and each contestant is eligible for at most one. The model boils down to a winner-take-all competition when $V_2 = 0$.

[15] The prize allocation $V_1 = \ldots = V_N$ is clearly suboptimal, because no effort can be elicited in the equilibrium.

2.1 Winner-selection Mechanism

We adopt the popularly studied multi-winner nested Tullock contest (Clark and Riis, 1996, 1998) to depict the winner-selection mechanism that allows for multiple prize recipients.

The multi-winner nested contest can conveniently be described as a sequential lottery process. For a given effort profile $\boldsymbol{e} := (e^1, \dots, e^N)$, a contestant i is picked as the recipient of the first prize, V_1, with a probability

$$p_1^i(\boldsymbol{e}) := \begin{cases} \frac{(e^i)^r}{\sum_{j\in\mathcal{N}}(e^j)^r}, & \text{if } \boldsymbol{e} \neq \boldsymbol{0}, \\ \frac{1}{N}, & \text{if } \boldsymbol{e} = \boldsymbol{0}, \end{cases}$$

which is equivalent to a standard winner-take-all Tullock contest. The parameter r indicates the discriminatory power of the contest technology. Recall that a contestant is eligible for only one prize. The recipient of the first prize is removed immediately from the pool of contestants eligible for the rest of the prizes, and a similar lottery picks the recipient of the second prize from the remaining candidates. The process is repeated until all prizes have been distributed.

To put this formally, let Ω^m, $m \in \{1, \dots, N\}$, be the set of contestants who remain eligible for the mth-draw—i.e., those who were not picked in the previous $m-1$ draws—with $\Omega^1 \equiv \mathcal{N}$. Further denote by $\boldsymbol{e}^{\Omega^m}$ the effort profile of all contestants in the set Ω^m, with $\boldsymbol{e}^{\Omega^1} \equiv \boldsymbol{e}$. The probability of a contestant i's receiving the mth prize *conditional* on his not having been picked in the previous $m-1$ draws is given by

$$p_m^i\left(\boldsymbol{e}^{\Omega^m}; \Omega^m\right) := \begin{cases} \frac{(e^i)^r}{\sum_{j\in\Omega^m}(e^j)^r}, & \text{if } \boldsymbol{e}^{\Omega^m} \neq \boldsymbol{0}, \\ \frac{1}{N-m+1}, & \text{if } \boldsymbol{e}^{\Omega^m} = \boldsymbol{0}, \end{cases} \tag{1}$$

Fu and Lu (2012b) demonstrate that the multi-winner nested Tullock contest model is underpinned by a unique noisy ranking system, with the standard single-prize Tullock contest being a special case, i.e., $V_1 > 0$ and $V_m = 0$ for all $m \in \{2, \dots, N\}$. Imagine that contestants are evaluated through a set of noisy signals of their performance y^is. Following the discrete choice framework of McFadden (1973, 1974),[16] the noisy signal y^i is assumed to be described by

$$\ln y^i = \ln f_i(e^i) + \varepsilon^i, \forall\, i \in \mathcal{N},$$

where the deterministic and strictly increasing production function $f_i(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ measures the output of contestant i's effort e^i,[17] and the additive noise term ε^i reflects the randomness

[16]The framework of McFadden's discrete choice model is further introduced and studied in various respects by works collected in Manski and McFadden (1981).

[17]Define $\ln f_i(e^i) = -\infty$ if $f_i(e^i) = 0$.

of the production process or the imperfection of the measurement and evaluation process. Idiosyncratic noises $\boldsymbol{\varepsilon} := (\varepsilon^1, \ldots, \varepsilon^N)$ are independently and identically distributed, being drawn from a type I extreme-value (maximum) distribution, with a cumulative distribution function

$$\Gamma(\varepsilon^i) = \exp\left[-\exp\left(-\varepsilon^i\right)\right], \varepsilon^i \in (-\infty, +\infty), \forall\, i \in \mathcal{N}.$$

A complete ranking among contestants immediately results when the shocks $\boldsymbol{\varepsilon} \equiv (\varepsilon^1, \ldots, \varepsilon^N)$ are realized, and each contestant is rewarded based on his rank, i.e., one receives a prize V_m if he is ranked in the mth place. Fu and Lu (2012b) verify that the probability of a contestant's being ranked in the mth place conditional on his failure to secure a more favorable rank coincides with the specification of Equation (1). That is, despite the literal resemblance to a sequential lottery process, the multi-winner nested contest model is uniquely underpinned by this noisy ranking system, which generates a complete ranking of contestants simultaneously for a given effort profile.

2.2 Contestants' Preference

Each contestant has an initial wealth $w \geq 0$. Contestants are assumed to be risk averse with a (weakly) concave Bernoulli utility function $u(\cdot)$ that satisfies the following conditions.

Assumption 1 ***(Risk-averse contestants)*** *Contestants' utility function $u(\cdot)$ is continuously differentiable and satisfies $u'(c) > 0$, and $u''(c) \leq 0$ for all $c \in \mathbb{R}_+$.*

We assume that effort is costly and reduces a contestant's wealth at a unitary price. Therefore, a contestant ends up with a wealth of $w+V_m-e^i$ if he wins the mth prize. Fixing an effort profile $\boldsymbol{e} \equiv (e^1, \ldots, e^N)$, denote by $P^i_m(\boldsymbol{e})$ a contestant i's ex ante probability of winning the mth prize. A contestant i's expected utility under a prize series $\boldsymbol{V} := (V_1, \ldots, V_N)$ is then given by

$$\sum_{m=1}^{N} \left[P^i_m(\boldsymbol{e}) \times u(w + V_m - e^i)\right].$$[18]

2.3 Contest Design

The primary objective of the paper is to explore the incentive effect of prize allocation in the contest. Prior to the competition, a contest designer splits a fixed prize purse of $V > 0$ into N nonnegative prizes with $V_1 \geq \ldots \geq V_N \geq 0$ and $\sum_{m=1}^{N} V_m \leq V$. She announces

[18] Alternatively, we can assume that effort is nonmonetary, and contestant's utility is given by $\sum_{m=1}^{N}[P^i_m(\boldsymbol{e}) \times u(w + V_m)] - c(e^i)$, where $c(\cdot)$ is the effort cost function with $c' > 0$ and $c'' \geq 0$. This case corresponds to the ability contest model of Schroyen and Treich (2016) and is widely adopted in tournament models. See Section 4.3.3 for more discussions of the differences between these two model specifications.

the prize schedule $\boldsymbol{V} \equiv (V_1, \ldots, V_N)$ publicly, which is thus commonly known to contestants when they exert their efforts. The designer aims to maximize the total effort of the contest, i.e., $\sum_{i=1}^{N} e^i$.

3 Equilibrium Analysis

In this section, we characterize the bidding equilibrium of the multi-prize contest game. As is well known in the contest literature, pure-strategy bidding dissolves when the contest is excessively discriminatory, i.e., when r is sufficiently large. Randomized bidding emerges, while the properties of equilibrium bidding strategies remain elusive in the literature.[19] For the sake of tractability, we focus on the case of moderate r. Specifically, let us define $\bar{r}$ as

$$\bar{r} := \min\left\{1\,,\ \frac{4}{\sum_{g=2}^{N}\frac{1}{g}}\right\}.$$[20]

We impose the following regularity condition on the contest technology:

Assumption 2 ***(Moderate discriminatory power)*** $0 < r \leq \bar{r}$.

We now characterize the equilibrium for an arbitrary fixed prize schedule $\boldsymbol{V} \equiv (V_1, \ldots, V_N)$. Following the convention of this literature (e.g., Fu and Lu, 2012a), we focus on symmetric interior pure-strategy equilibria in which all contestants exert the same amount of effort. To search for the equilibrium, we use the symmetric opponents form approach (SOFA)[21] and assume that all players other than one indicative player place the same bid e. An effort e' allows the indicative contestant to win the mth prize with a probability

$$P_m\left(e', e\right) \equiv \frac{(N-1)!}{(N-m)!} \times \left(\prod_{j=1}^{m-1} \frac{(e)^r}{(N-j)(e)^r + (e')^r}\right) \times \frac{(e')^r}{(N-m)(e)^r + (e')^r},$$

where the term $\frac{(e')^r}{(N-m)(e)^r+(e')^r}$ is the probability of his being picked in the mth draw conditional on that he has not been selected for the previous $m-1$ prizes. It is straightforward to verify that $P_m(e, e) = 1/N$.

The indicative contestant chooses his effort e' for the following expected utility maxi-

[19] The reader is referred to Ewerhart (2015) for an analysis of a winner-take-all Tullock contest for large but finite r.

[20] It can be verified that $4/[\sum_{g=2}^{N} 1/g] > 1$ for $N \leq 82$, implying that $\bar{r} = 1$ for $N \leq 82$.

[21] See Hefti (2017) and Drugov and Ryvkin (2018).

mization problem:

$$\max_{e'\geq 0} \pi(e',e) := \sum_{m=1}^{N} \Big[P_m\left(e',e\right) \times u\left(w+V_m-e'\right)\Big]. \tag{2}$$

The first-order condition with respect to e' leads to

$$\sum_{m=1}^{N}\left[\frac{\partial P_m\left(e',e\right)}{\partial e'} \times u\left(w+V_m-e'\right)\right] = \sum_{m=1}^{N}\Big[P_m\left(e',e\right)\times u'\left(w+V_m-e'\right)\Big].$$

A symmetric equilibrium requires $e'=e$, in which case

$$\left.\frac{\partial P_m\left(e',e\right)}{\partial e'}\right|_{e'=e} = \frac{r}{Ne}\times\left(1-\sum_{g=0}^{m-1}\frac{1}{N-g}\right) =: \mu_m \times \frac{r}{Ne},$$

where μ_m, with $m\in\{1,\ldots,N\}$, is defined as

$$\mu_m := 1-\sum_{g=0}^{m-1}\frac{1}{N-g}.$$

It is straightforward to verify that $\mu_1>\mu_2>\ldots>\mu_N$ and $\sum_{m=1}^{N}\mu_m=0$. Therefore, the term $\partial P_m(e',e)/\partial e'\big|_{e'=e}$ strictly decreases with m. To put this intuitively, additional effort affords him a higher probability of obtaining a better prize, and equivalently, renders him less likely to fall behind and end up with a less lucrative prize. The following condition must hold for an interior symmetric pure-strategy equilibrium:

$$\frac{r}{Ne}\times\sum_{m=1}^{N}\left[\mu_m\times u(w+V_m-e)\right] = \frac{1}{N}\times\sum_{m=1}^{N}u'(w+V_m-e). \tag{3}$$

We further impose the following requirement on contestants' preference.

Assumption 3 ***(NIARA preferences)*** *Contestants' utility function exhibits nonincreasing absolute risk aversion (NIARA), i.e., $-u''(c)/u'(c)$ is nonincreasing in c.*

The assumption is first proposed by Arrow (1970). The NIARA condition is satisfied by a broad spectrum of utility functions, such as the familiar constant absolute risk aversion (CARA, henceforth) and constant relative risk aversion (CRRA, henceforth) utility functions. A plethora of experimental and empirical findings provide evidence for the prevalence of decreasing absolute risk aversion (see, for instance, Friend and Blume, 1975). It is worth noting that NIARA implies $u'''\geq 0$, i.e., contestants are prudent.[22] The following result is

[22]To see this, note that $\frac{d}{dc}\left(-\frac{u''(c)}{u'(c)}\right)\leq 0$ is equivalent to $u'''(c)\geq[-u''(c)]^2/u'(c)\geq 0$.

obtained.

Theorem 1 ***(Equilibrium existence and uniqueness under NIARA preferences)*** *Suppose that Assumptions 1, 2, and 3 are satisfied. Then there exists a unique symmetric pure-strategy equilibrium of the contest game, in which each contestant's equilibrium effort is the solution to Equation* (3)*.*

Theorem 1 shows that the symmetric effort profile that solves Equation (3) constitutes the unique symmetric equilibrium of the game when the mild requirement of Assumptions 1-3 are satisfied. Technically speaking, the assumption of NIARA is imposed to guarantee that contestants' payoff $\pi(e', e)$ is concave in his effort entry e'[23], and thus the first-order condition $\partial\pi(e', e)/\partial e' = 0$ is not only a necessary but also a sufficient condition to characterize a contestant's best response. Furthermore, the NIARA preference ensures that Equation (3) has a unique solution for $r \leq \bar{r}$.

It is worth noting that Assumption 1 imposes a conservative upper bound on the size of r for the existence of pure-strategy equilibrium. The upper bound can be relaxed under a broad array of utility functions.[24] Consider, for instance, the usual CARA preferences.

Corollary 1 ***(Equilibrium existence and uniqueness under CARA preferences)*** *Suppose that contestants exhibit CARA. There exists a unique symmetric pure-strategy equilibrium of the contest game for all $r \leq 1$.*

The literature also popularly assumes quadratic (mean-variance) preferences to model risk aversion, in which case contestants have a utility function

$$u(c) = c - \frac{\gamma}{2}c^2, \text{ with } \gamma > 0 \text{ and } c \leq 1/\gamma.$$

Assumption 3 is violated, and hence Theorem 1 does not apply. We next establish a sufficient condition for the existence and uniqueness of symmetric equilibrium in this scenario.

Theorem 2 ***(Equilibrium existence and uniqueness under quadratic preferences)*** *Suppose that contestants have quadratic utility and $u'(w + V) \geq 0$. There exists a unique symmetric pure-strategy equilibrium of the contest game for all $r \leq \frac{1}{2(N-1)^2+1}$.*

The upper bound $\frac{1}{2(N-1)^2+1}$ ensures a concave expected payoff function $\pi(e', e)$. Although a formal proof is absent, simulations show that concavity is preserved as long as $r \leq 1$, indicating that the symmetric pure-strategy equilibrium remains unique more generally.

[23]Note that $P_m(e', e)$ is not necessarily concave in contestants' effort entry e when two or more prizes are distributed in a contest. Therefore, the concavity of the expected payoff function is not obvious, even if contestants are risk neutral and all opponents employ the same bidding strategy. See Fu, Wu and Zhu (2019) for more details.

[24]Simulations show that the upper bound for r can be relaxed to one with CRRA preferences.

4 Contest Design: Prize Allocation

The equilibrium analysis allows us to formally explore the optimal prize allocation problem. The contest designer chooses the prize schedule $\boldsymbol{V} \equiv (V_1, \ldots, V_N)$ prior to the competition, anticipating that contestants play the symmetric equilibrium characterized by (3). The condition (3) reveals the fundamental trade-off faced by a contestant when choosing his effort strategy, which is critical to understanding how a variation in prize schedule could affect contestants' bidding incentives. Recall that the condition is written as

$$\underbrace{\frac{r}{Ne} \times \sum_{m=1}^{N} \left[\mu_m \times u(w + V_m - e)\right]}_{\text{marginal benefit of effort}} = \underbrace{\frac{1}{N} \times \sum_{m=1}^{N} u'(w + V_m - e)}_{\text{marginal cost}}.$$

The left-hand and right-hand sides of Equation (3), respectively, represent the marginal benefit and marginal cost of a contestant's effort in terms of his utility gain/loss evaluated in the symmetric equilibrium. A higher effort varies the probability distribution of all possible outcomes. For each possible outcome, i.e., being ranked in an arbitrary place m, the marginal impact of effort on his utility through this channel is given by $\partial P_m(e', e)/\partial e'\big|_{e'=e} \times u(w + V_m - e')$; summing up over all possible outcomes, the overall marginal effect boils down to the left-hand side of Equation (3) by the fact that $\partial P_m(e', e)/\partial e'\big|_{e'=e} = \mu_m \times \frac{r}{Ne}$.

An increase in effort, however, also consumes the contestant's wealth and generates disutility in all possible outcomes. The marginal effect is captured by the right-hand side of Equation (3). The marginal disutility, $u'(w + V_m - e)$, is state-dependent, and the overall marginal cost is obtained by summing up $u'(w + V_m - e)$ over all states in a symmetric equilibrium.

A variation in the prize allocation profile compels a contestant to rebalance between the marginal benefit of effort and marginal cost. This could increase equilibrium effort if it (i) increases each contestant's marginal benefit of effort, and/or (ii) decreases the marginal cost. Contestants' risk attitude plays a critical role in contestants' trade-off.

Let us first consider a benchmark case of risk-neutral contestants. In this case, the left-hand side of Equation (3) is

$$\frac{r}{Ne} \times \sum_{m=1}^{N} \left[\mu_m \times u(w + V_m - e)\right] = \frac{r}{Ne} \times \left[\sum_{m=1}^{N} \mu_m \times V_m\right],$$

while the right-hand side simply boils down to one, which is a constant and independent of the prevailing prize structure. Varying prize allocation affects only the marginal benefit of effort but not the marginal cost.

With risk-neutral contestants, maximizing the marginal benefit of effort, i.e., maximizing the sum $\sum_{m=1}^{N} \mu_m \times V_m$, simply requires concentrating the entire prize purse on the top prize V_1. To see this, imagine that a small prize money ϵ is shifted from V_m to V_{m+1}. With linear utility, the change to the marginal benefit boils down to $-\frac{r}{Ne}(\mu_m - \mu_{m+1})\epsilon$, which is strictly negative because μ_m is strictly decreasing with m: The entire prize purse must be concentrated on the top prize, as it incentivizes contestants more than any lower-rank prize. This logic underpins the winner-take-all result of Fu and Lu (2012a) and Schweinzer and Segev (2012) in contests with risk-neutral contestants.

Risk aversion, however, triggers the two additional effects mentioned in Section 1. First, by Equation (3), when prize money is shifted between prizes, the change in utility is nonlinear. Concentrating the entire prize purse on V_1 does not necessarily maximize the marginal benefit of effort, $\frac{r}{Ne} \times \sum_{m=1}^{N} \left[\mu_m \times u(w + V_m - e)\right]$, because of the nonlinearity of $u(\cdot)$, despite the strictly decreasing μ_m. Second, varying the prize schedule affects not only the marginal benefit of effort but also the marginal cost: $\frac{1}{N} \times \sum_{m=1}^{N} u'(w + V_m - e)$ may depend on the particular prize structure.[25]

Imagine, again, a hypothetical shift of a very small amount of prize money ϵ from V_1 to V_2. Consider first the impact on marginal benefit. A direct loss results because of decreasing μ_m. The loss in $u(w+V_1-e)$, however, can be (at least partly) compensated by a larger gain in $u(w + V_2 - e)$ because of the contestant's concave utility function. The overall change in the marginal benefit amounts to $\frac{r}{Ne} \times \{\mu_2[u(w+V_2+\epsilon-e) - u(w+V_2-e)] - \mu_1[u(w+V_1-e) - u(w+V_1-\epsilon-e)]\}$, which boils down to $\frac{r}{Ne} \times [\mu_2 u'(w+V_2-e) - \mu_1 u'(w+V_1-e)]\epsilon$ for small ϵ. The direction of the change depends on the curvature of the utility function $u(\cdot)$, i.e., its second-order property.

We next consider its impact on marginal cost, i.e., $\frac{1}{N} \times \sum_{m=1}^{N} u'(w + V_m - e)$. Because of decreasing marginal utility, the shift of prize money leads to a hike of marginal disutility in the state of achieving the top rank, i.e., $u'(w + V_1 - e) < u'(w + V_1 - \epsilon - e)$; this, however, reduces the marginal disutility in the state of achieving the second rank, i.e., $u'(w + V_2 - e) > u'(w + V_2 + \epsilon - e)$. The direction of the overall change in the marginal cost, $\frac{1}{N} \times [|u''(w + V_1 - e)| - |u''(w + V_2 - e)|]\epsilon$, depends on the curvature of the marginal utility function $u'(\cdot)$, i.e., the third-order property of the utility function $u(\cdot)$. Prudent contestants—with a positive $u'''(\cdot)$—implies that $\left|u''(w + V_1 - e)\right| - \left|u''(w + V_2 - e)\right| < 0$: They are more sensitive to downward risk because $\left|u''(\cdot)\right|$ is decreasing, so they perceive more significant disutility for a given amount of forgone effort when they end up with (the small) V_2 than they do if they end up with (the large) V_1. The shift in prize money alleviates the pain and reduces marginal cost of effort.

[25] One exception is the quadratic utility function with linear marginal cost u', or equivalently, $u''' = 0$. It is straightforward to verify that the marginal cost curve depends on the designer's whole budget V in this case.

The former effect alludes to the usual preference for a smoother consumption profile across different states caused by risk aversion, whereas the second is analogous to that underlying the precautionary saving motive (Kimball, 1990) and self-protective behavior (Dachraoui, Dionne, Eeckhoudt and Godfroid, 2004) caused by prudence. An intuitive account of optimal prize allocation can be immediately obtained in light of the discussion laid out above.

1. Multiple prizes are more likely to emerge in the optimum when contestants exhibit a higher degree of risk aversion. The preference for a smoother consumption profile across different states implies that shifting prize money to lower-rank prizes may increase the marginal benefit of effort.

2. Multiple prizes tend to be more appealing when u' is convex, i.e., when contestants are *prudent*, with $u'''(\cdot) \geq 0$. A more dispersed prize allocation profile effectively reduces contestants' pain in less favorable states, which reduces the marginal cost of effort.

Our subsequent results are all interpreted in light of the above rationale. Before we proceed, let us lay out some preliminaries. Define

$$\tau := \max\left\{\sup\left\{-\frac{u'''(c)}{u''(c)}\,\middle|\, w - \frac{V}{N} < c < w + V\right\}, 0\right\}, \tag{4}$$

which is useful for our analysis. The term $-u'''(c)/u''(c)$ in expression (4) is referred to as the coefficient of absolute prudence in the economics literature (see Kimball, 1990). If $-u'''(c)/u''(c) > 0$ holds globally, then an individual's marginal utility is convex in his consumption. In words, τ is the maximum absolute prudence level of a contestant in his relevant support of wealth.[26]

Further, the following property of the optimum can readily be obtained.

Proposition 1 ***(Consecutive and monotone prize series)*** *If $V_j > 0$ for some $j \in \{2, \ldots, N\}$ in the optimal contest, then $V_{j-1} > V_j$.*

Proposition 1 states that positive prizes are never equal in the optimum despite contestants' risk aversion, and a higher rank is always rewarded more. As shown in our proof, a properly set prize premium for a higher rank increases the marginal benefit of effort and incentivizes contestants.

[26] To see this more clearly, note that the equilibrium effort level cannot exceed $\frac{V}{N}$. Therefore, a representative contestant's income in any state, i.e., $w + V_m - e$, is bounded from above by $w + V$ and bounded below by $w - \frac{V}{N}$.

4.1 Optimality of Single-prize Contests

We first provide a sufficient condition under which the usual winner-take-all contest emerges in the optimum.

Proposition 2 ***(Optimality of single-prize contests)*** *Suppose that the contest game has a unique symmetric pure-strategy equilibrium and $N \geq 3$. Then a single-prize contest generates a larger amount of total effort than any multiple-prize contest if*

$$\frac{u'\left(w - \frac{V}{N}\right)}{u'\left(w + V\right)} \leq \frac{r\mu_1 + \frac{V}{N}\tau}{r\mu_2 + \frac{V}{N}\tau}. \tag{5}$$

Although the condition required in Proposition 2 is sufficient but not necessary for the optimality of single-prize contests, it yields useful economic implications. First, the condition is more likely to be satisfied when $u'(w - \frac{V}{N})/u'(w + V)$ remains moderate. This occurs when $u'(\cdot)$ decreases more gradually, in which case a contestant's preference for a smoother consumption profile is not excessively strong: That is, a low-degree risk aversion is in place. When prize money is shifted from a higher-rank prize to a lower-rank one, the increased marginal benefit caused by a smoother wealth distribution across states is less likely to counteract the negative effect caused by decreasing μ_m.

Second, the right-hand side of the condition, $(r\mu_1 + \frac{V}{N}\tau)/(r\mu_2 + \frac{V}{N}\tau)$, strictly decreases with the degree of prudence τ. Therefore, the condition is also more likely to be satisfied when contestants are not excessively prudent: That is, when prize money is shifted to a lower-rank prize, a contestant's marginal cost does not decrease substantially.

Clearly, the above condition automatically holds for linear utility functions—i.e., with risk-neutral contestants: The left-hand side of the condition boils down to one, while the right-hand side is always strictly greater than one. Clark and Riis (1996, 1998), Schweinzer and Segev (2012), and Fu and Lu (2012a) establish the optimality of single-prize contests in the case of symmetric and risk-neutral contestants. Proposition 2 extends the boundary for these winner-take-all results, as it allows for weak risk aversion and/or weak prudence. We illustrate this notion by using the convenient CARA, CRRA, and quadratic utility functions.

Example 1 ***(CARA utility)*** *Suppose that the utility function takes the form*

$$u(c) = 1 - \exp(-\alpha c), \textit{ with } \alpha > 0.$$

It is straightforward to verify that $-u'''/u'' = \alpha$ and hence $\tau = \alpha$. The condition $\frac{u'\left(w - \frac{V}{N}\right)}{u'(w+V)} \leq$

$\frac{r\mu_1+\frac{V}{N}\tau}{r\mu_2+\frac{V}{N}\tau}$ *can be equivalently written as*

$$\exp\left(\frac{N+1}{N}\alpha V\right) \leq \frac{r\mu_1+\frac{V}{N}\alpha}{r\mu_2+\frac{V}{N}\alpha}.$$

Note that the left-hand side and the right-hand side of the above inequality, respectively, approach 1 *and* $\frac{\mu_1}{\mu_2} > 1$*, as* $\alpha \searrow 0$*, which in turn indicates that the above condition holds for sufficiently small* α*.*

Example 2 ***(CRRA utility)*** *Suppose that the utility function takes the form*

$$u(c) = \begin{cases} (c^{1-\beta}-1)/(1-\beta), & \text{if } \beta > 0, \text{ and } \beta \neq 1, \\ \ln(c), & \text{if } \beta = 1. \end{cases}$$

Simple algebra can verify that $\tau = (1+\beta)/(w-\frac{V}{N})$*, and that* $\frac{u'\left(w-\frac{V}{N}\right)}{u'(w+V)} \leq \frac{r\mu_1+\frac{V}{N}\tau}{r\mu_2+\frac{V}{N}\tau}$ *holds for sufficiently small* β*.*

Example 3 ***(Quadratic utility)*** *Suppose that the utility function takes the form*

$$u(c) = c - \frac{\gamma}{2}c^2, \text{ with } \gamma > 0 \text{ and } c \leq 1/\gamma.$$

It is evident that $u'''(c) = 0$ *and thus* $\tau = 0$*. Moreover,* $\frac{u'\left(w-\frac{V}{N}\right)}{u'(w+V)} \leq \frac{r\mu_1+\frac{V}{N}\tau}{r\mu_2+\frac{V}{N}\tau}$ *is equivalent to* $\frac{1-\gamma(w-\frac{V}{N})}{1-\gamma(w+V)} \leq \frac{\mu_1}{\mu_2}$*, which, again, holds when* γ *becomes sufficiently small.*

We subsequently demonstrate that multiple prizes would emerge as the optimum when stronger risk aversion and prudence are present.

4.2 Optimality of Multiple-prize Contests

We now derive the condition under which the winner-take-all principle fades away. For notational convenience, denote by e_s the equilibrium effort level in a single-prize contest, i.e., under a prize schedule $\boldsymbol{V}_s := (V, 0, \ldots, 0)$. A sufficient condition for the optimality of multi-prize contests is laid out below.

Proposition 3 ***(Optimality of multiple-prize contests)*** *Suppose that the contest game has a unique symmetric pure-strategy equilibrium and* $N \geq 3$*. A multiple-prize contest generates more total effort than the single-prize contest—i.e.,* $\boldsymbol{V}_s := (V, 0, \ldots, 0)$*— if* $u''' \geq 0$ *and*

$$\frac{u'(w-e_s)}{u'(w+V-e_s)} > \frac{\mu_1}{\mu_2}.$$

Proposition 3 states that a multiple-prize contest can be optimal with risk-averse and weakly prudent contestants. Obviously, the condition $\frac{u'(w-e_s)}{u'(w+V-e_s)} > \frac{\mu_1}{\mu_2}$ degenerates to $1 > \frac{\mu_1}{\mu_2}$ under risk neutrality and can never be satisfied.

We conduct the following thought experiment to interpret the conditions. Begin with a single-prize contest and suppose that the designer shifts the prize money and awards a small second prize of size $\epsilon > 0$. We demonstrate that the reallocation of prize money can effectively incentivize efforts when contestants are sufficiently risk averse and prudent. In the single-prize contest, one ends up with a wealth of $w + V - e_s$ if he wins and $w - e_s$ if he loses. When a small prize for the runner-up is introduced, the overall impact on the marginal benefit of effort amounts to $[\mu_2 u'(w - e_s) - \mu_1 u'(w + V - e_s)]\epsilon$ for small ϵ. Strong risk aversion implies a relatively larger utility gain for a wealth increase in less favorable states. As a result, $\big[\mu_2 u'(w - e_s) - \mu_1 u'(w + V - e_s)\big]$ can be positive, as the gain from a smoother consumption profile overshadows the incentive loss caused by the less rewarding top prize. This occurs when $u'(w - e_s)/u'(w + V - e_s) > \mu_1/\mu_2$.

Further, recall that the marginal cost of effort is represented by the expected marginal utility $\frac{1}{N} \times \sum_{m=1}^{N} u'(w + V_m - e)$. When contestants are (weakly) prudent (i.e., $u''' \geq 0$), the marginal utility is (weakly) convex. Awarding a second prize reduces the additional harm caused by a contestant's nonrefundable input in the less favorable state (i.e., being ranked in the second place) more than it increases that in the favorable state (i.e., achieving the top rank). The sum decreases accordingly, and a lower marginal effort cost would further incentivize the contestant.

In summary, the conditions $u'(w - e_s)/u'(w + V - e_s) > \mu_1/\mu_2$ and $u''' \geq 0$ imply that the hypothetical shift in prize money strictly increases marginal benefit, due to strong risk aversion, but not marginal cost, due to prudence. This renders the winner-take-all contest suboptimal.

Obviously, the conditions established in Proposition 3, as well as those in Proposition 2, are sufficient but not necessary. The ratio $u'(w-e_s)/u'(w+V-e_s)$ in Proposition 3 is strictly less than $u'(w - \frac{V}{N})/u'(w + V)$ in Proposition 2, while the ratio μ_1/μ_2 in Proposition 3 is larger than $(r\mu_1 + \frac{V}{N}\tau)/(r\mu_2 + \frac{V}{N}\tau)$ in Proposition 2 when contestants are prudent. Therefore, it is possible to have $u'(w - e_s)/u'(w + V - e_s) < \mu_1/\mu_2$ and $u'(w - \frac{V}{N})/u'(w + V) > (r\mu_1 + \frac{V}{N}\tau)/(r\mu_2 + \frac{V}{N}\tau)$ simultaneously. Neither set of conditions applies in this case. The comparison of single prize vis-à-vis multiple prizes remains elusive in general: Nontrivial risk aversion, possibly coupled with prudence, casts doubt on the optimality of single prize, while the intensity of risk aversion is insufficient to ensure the optimality of multiple prizes. The comparison may sensitively depends on the properties of the particular utility function. In Section 4.3, we demonstrate with examples that prudence could effectively complement risk aversion to support multiple prizes even when the latter is relatively weak.

A closer look at Proposition 3 further yields the following.

Corollary 2 *Suppose that contestants are prudent (i.e., $u''' > 0$) and a unique symmetric pure-strategy equilibrium exists. Awarding multiple prizes is optimal to an effort-maximizing contest designer if (i) V is sufficiently large and $u'(\infty) < \frac{\mu_2}{\mu_1 u'(w)}$; or (ii) N is sufficiently large.*

When V is large, a winner-take-all prize structure renders the contest riskier because of the more polarized wealth distribution across states. The spread between $u'(w - e_s)$ and $u'(w + V - e_s)$ enlarges for large V, which makes more likely the condition $\frac{u'(w-e_s)}{u'(w+V-e_s)} > \frac{\mu_1}{\mu_2}$: A concave utility function could severely discount the marginal gain from a large prize. When the contest involves a larger number of contenders, one expects a smaller likelihood of achieving the top rank. Lower-rank prizes gain in their appeal, as they provide contestants with additional avenues to reward, which can in turn increase the marginal benefit of effort.

Proposition 3 and Corollary 2 provide sufficient conditions for multiple prizes to be optimal. However, the exact form of the optimal prize schedule can be obtained only in specific setting. However, we are able to establish an upper bound on the optimal number of prizes. Recall that

$$\tau := \max\left\{\sup\left\{-\frac{u'''(c)}{u''(c)}\,\middle|\, w - \frac{V}{N} < c < w + V\right\}, 0\right\}.$$

Further define

$$N_p := \max\left\{j = 1, \ldots, N \,\middle|\, r \times \mu_j + \frac{V}{N}\tau > 0\right\}. \tag{6}$$

The following can be obtained.

Proposition 4 ***(Upper bound of optimal number of prizes)*** *The number of positive prizes in the optimal contest is no greater than N_p.*

Again, the logic can be understood in light of the rationale outlined above. Suppose that the contest designer considers setting aside a small additional prize indexed by j. The move would affect both the marginal benefit and marginal cost of effort, as previously discussed. Recall that the marginal benefit of effort is given by $\frac{r}{Ne} \times \sum_{m=1}^{N}[\mu_m \times u(w + V_m - e)]$ from Equation (3). The impact of the jth prize on the marginal benefit is thus proportional to μ_j. Note that the effect can be negative, depending on the sign of μ_j: Introducing a lower-rank prize softens the competition and weakens contestants' incentives to vie for more favorable positions. Further, the marginal cost of effort is given by $\frac{1}{N} \times \sum_{m=1}^{N} u'(w + V_m - e)$. It can be reduced if they are prudent, while the magnitude of the reduction depends on the degree of prudence, i.e., τ. The term $r \times \mu_j + \frac{V}{N}\tau$ in expression (6) thus captures the *net* effect on effort incentive of the small additional prize j. Awarding such a prize is suboptimal if the net effect turns negative. It is noteworthy that by this rationale, a low-rank prize is likely to emerge in the optimum even if it creates a direct negative incentive and reduces marginal benefit of effort, i.e., $\mu_j < 0$: It could alleviate the pain caused by effort, which makes up for the loss in marginal benefit. We illustrate this possibility in Section 4.3.3.

4.3 Discussion

Risk aversion and prudence are two highly related concepts used to describe economic agents' risk attitude. When the prize structure varies, the former determines its effect on marginal benefit of effort, while the latter influences the marginal cost. Next, we discuss in further detail the role played by risk aversion and prudence in determining the optimum. Propositions 1 to 4 provide a partial characterization of the contest designer's optimal prize schedule. The specific form of optimal prize allocation depends sensitively on the property of contestants' utility function. To gain additional mileage, we impose more structures on the utility function. We first follow Cornes and Hartley (2012, Example 1) and consider a quadratic utility function with $u'''(\cdot) = 0$: The effect of the utility functions' third-order property is entirely muted, which allows us to abstract away the role played by prudence and focus on that of risk aversion. We then consider a CARA utility function, which features a constant absolute prudence. A comparison to the case of quadratic utility highlights the role played by prudence. Finally, we consider a variation to our model. We consider utility functions that are separable in one's income and effort cost, and discuss the implications of the modeling nuance.

4.3.1 Role of Risk Aversion

We now shed some light on the impact of contestants' risk preferences on the optimal prize structure in a context of quadratic utility function. Assume that the utility function takes the form

$$u(c) = c - \frac{\gamma}{2}c^2,$$

where $\gamma \geq 0$ measures the degree of contestants' risk aversion. Its third-order derivative reduces to zero. As a result, when prize money is being shifted between prizes of different ranks, contestants' marginal cost of effort remains constant. Therefore, the change in effort incentive is solely determined by the change in marginal benefit, which depends on contestants' risk aversion.

Recall that Propositions 2 and 3 provide sufficient but not necessary conditions for the optimality of single prize and multiple prizes, respectively. The optimum remains ambiguous in general when neither condition is satisfied, in which case, as discussed above, the optimum depends on the aforementioned three-way trade-off, and both risk aversion and prudence may play a role. For quadratic utility, the ambiguity fades away because the parameter $\tau = -u'''(\cdot)/u''(\cdot)$, a contestant's absolute prudence, reduces to zero. We obtain the following result, by which a single-prize contest emerges as the optimum once the condition in Proposition 3—which establishes the optimality of multi-prize contests—fails to hold.

Corollary 3 ***(Optimal prize allocation with quadratic utility)*** *Suppose that each contestant has a quadratic utility function. The effort-maximizing contest sets multiple prizes if $\frac{u'(w-e_s)}{u'(w+V-e_s)} > \frac{\mu_1}{\mu_2}$ and a single prize if $\frac{u'(w-e_s)}{u'(w+V-e_s)} < \frac{\mu_1}{\mu_2}$.*

The specific setting of quadratic utility allows us to further examine two main questions: (i) How would the optimal number of prizes change as contestants become more risk averse? (ii) How would the distribution of prizes change as contestants become more risk averse? Recall that the parameter γ measures the degree of risk aversion. We have the following observations.

Example 4 ***(Comparative statics of optimal prize allocation in risk aversion)*** *Suppose that $u(c) = c - \frac{\gamma}{2}c^2$, $N = 10$, $r = 1$[27], $w = 0$, and $V = 1$. By expression (6), the maximum number of positive prizes is 6 and thus we must have $V_7^* = \ldots = V_{10}^* = 0$ in the optimal contest.*[28] *The equilibrium effort e^* and optimal prize series $(V_1^*, \ldots, V_6^*)$ for different values of γ are reported as follows:*

γ	e^*	V_1^*	V_2^*	V_3^*	V_4^*	V_5^*	V_6^*
$\gamma_1 = 0.1$	0.0864	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\gamma_2 = 0.3$	0.0816	0.6918	0.3082	0.0000	0.0000	0.0000	0.0000
$\gamma_3 = 0.5$	0.0786	0.5812	0.3702	0.0486	0.0000	0.0000	0.0000
$\gamma_4 = 0.7$	0.0765	0.4997	0.3581	0.1422	0.0000	0.0000	0.0000
$\gamma_5 = 0.9$	0.0748	0.4544	0.3514	0.1942	0.0000	0.0000	0.0000

The first pattern to notice is that the contest designer would (weakly) increase the number of prizes as contestants become more risk averse, i.e., when γ increases. The observation is intuitive. Contestants prefer a smoother consumption profile as γ ascends. The contest designer would set aside more prizes accordingly, because this enlarges the marginal benefit

[27] As previously mentioned in the main text, simulation shows that contestants' expected utility is concave in his effort entry for $r \leq 1$, and thus a unique symmetric pure-strategy equilibrium exists.

[28] Note that $\mu_6 = 1 - \sum_{g=0}^{5} \frac{1}{10-g} \approx 0.1544 > 0$ and $\mu_7 = 1 - \sum_{g=0}^{6} \frac{1}{10-g} \approx -0.0956 < 0$. Therefore, $N_p = 6$.

of effort. The second pattern to notice is that the distribution of prizes in the optimum tends to be more even as γ increases. Specifically, as γ increases from 0.5 to 0.9, the optimal number of prizes remains the same; however, the contest designer reallocates the prize money by reducing the sizes of prizes for higher ranks and increasing those for lower ranks. Both observations demonstrate that a higher degree of risk aversion renders prize allocation more dispersed over different ranks.

4.3.2 Role of Prudence

We now consider the case of CARA utility. That is, each contestant has a utility function

$$u(c) = 1 - \exp(-\alpha c), \text{ with } \alpha > 0.$$

An economic agent with CARA utility must also be strictly prudent, and has a constant level of absolute prudence $\tau = -u'''(c)/u''(c) = \alpha$. With both risk aversion and prudence in place, a variation in prize allocation affects both the marginal benefit and the marginal cost of effort. Recall that by Proposition 3, a multi-prize contest outperforms a single-prize one if $u'(w - e_s)/u'(w + V - e_s) > \mu_1/\mu_2$ and $u'''(\cdot) \geq 0$. When prize money is shifted from a single top prize to a second prize, the former condition ensures that sufficient risk aversion counteracts the superior incentive provided by top prize and magnifies marginal benefit. (Weak) prudence only ensures that marginal cost of effort would not increase. One may conjecture that with strictly prudent contestants, multiple prizes can be optimal even if the condition established in Proposition 3 is not met, in which a second prize does not amplify marginal benefit, but marginal cost strictly decreases. With CARA utility, we obtain the following result.

Corollary 4 ***(Optimal prize allocation with CARA utility)*** *Suppose that contestants' utility exhibits constant absolute risk aversion of $\alpha > 0$. The effort-maximizing contest sets multiple prizes if $\frac{u'(w-e_s)}{u'(w+V-e_s)} > \frac{2\mu_1}{\mu_1[1-\exp(-\alpha V)]+\mu_2[1+\exp(-\alpha V)]}$.*

It is straightforward to verify that the right-hand side of the above condition, $2\mu_1/\{\mu_1[1-\exp(-\alpha V)] + \mu_2[1+\exp(-\alpha V)]\}$, strictly falls below the ratio μ_1/μ_2. Multiple prizes emerge in the optimum even if a shift of prize money from a single top prize to the second prize reduces marginal benefit of effort. Prudence plays a nontrivial role by decreasing marginal cost of effort. This complements the effect of risk aversion in catalyzing the optimum with multiple prizes.

4.3.3 Nonseparable Utility vs. Separable Utility

Glazer and Hassin (1988), Krishna and Morgan (1998), Akerlof and Holden (2012), and

Drugov and Ryvkin (2019) assume that individual's utility is additive and separable in her income and cost of effort, and the utility function in expression (2) becomes

$$\sum_{m=1}^{N} \left[P_m(e', e) \times u(w + V_m) \right] - e'.$$

Schroyen and Treich (2016) interpret such a setting as an account of an "ability contest" in which contestants engage in nonmonetary efforts to compete for monetary prizes. With a separable utility function in place, the symmetric pure-strategy equilibrium is given by

$$\frac{r}{Ne} \times \sum_{m=1}^{N} \left[\mu_m \times u(w + V_m) \right] = 1. \tag{7}$$

In contrast to Equation (3), a contestant's marginal cost of effort is independent of the prize schedule with additive and separable utility functions. In other words, a variation in the prize structure affects only the marginal benefit of effort, while the aforementioned effect triggered by contestants' prudence on marginal cost of effort would disappear.

Intuitively, the optimum no longer factors in the third-order property of the utility function, which nullifies the role played by prudence. An additional prize boosts the performance of the contest only if it enlarges the marginal benefit. Our analysis thus sheds light on this setting, and the results can readily be adapted. Proposition 1 is preserved under separable utility: It espouses a consecutive and monotone prize series for the optimum, as the monotonicity serves to improve marginal benefit of effort regardless. Results in parallel to Propositions 2 and 3 can also be established immediately by removing their requirement regarding prudence.

To better illustrate the implications of (non)separability (and prudence), we take a closer look at prize allocation under this alternative setup. Define N_p^s as

$$N_p^s := \max \left\{ j = 1, \ldots, N \,\middle|\, \mu_j > 0 \right\}. \tag{8}$$

The following result can be obtained:

Proposition 5 ***(Maximum number of prizes under separable utility)*** *The number of positive prizes in the optimal contest is no greater than N_p^s.*

Recall that the optimal number of prizes cannot exceed N_p in the case of nonseparable utility, where N_p is given by

$$N_p := \max\left\{ j = 1, \ldots, N \,\middle|\, r \times \mu_j + \frac{V}{N}\tau > 0 \right\}.$$

It is straightforward to observe that the upper bound on the number of prizes N_p degenerates to N_p^s when the measure of contestants' prudence—i.e., τ—reduces to zero. Under separable utility, the role of prudence fades away, and thus the condition would no longer involve τ.

Proposition 5, together with Proposition 4, unveils the implications of separability in contestants' utility function on prize allocation. Proposition 5 indicates that a larger number of prizes can be awarded in the optimum under nonseparable utility vis-à-vis under separable utility. In the latter case, a positive prize V_j can be desirable only if it contributes to contestants' marginal benefit of effort, i.e., $\mu_j > 0$. This requirement, however, can be relaxed under nonseparable utility, as a prize for low rank may serve to reduce marginal effort cost, despite the direct negative incentive it creates. The following example demonstrates such a possibility.

Example 5 ***(Nonseparable utility vs. separable utility)*** *Suppose that $u(c) = 1 - \exp(-\alpha c)$, $N = 5$, $r = 1$, $w = 0$, and $V = 1$. By expression (8), the maximum number of positive prizes $N_p^s = 3$ independent of contestants' degree of risk aversion α. The optimal prize series $(V_1^*, \ldots, V_5^*)$ for different values of α and model specifications are reported as follows:*

α	*utility*	N_p *or* N_p^s	*number of prizes*	V_1^*	V_2^*	V_3^*	V_4^*	V_5^*
$\alpha_1 = 1$	*nonseparable*	$N_p = 3$	2	0.6568	0.3432	0.0000	0.0000	0.0000
$\alpha_1 = 1$	*separable*	$N_p^s = 3$	2	0.6873	0.3127	0.0000	0.0000	0.0000
$\alpha_2 = 3$	*nonseparable*	$N_p = 4$	3	0.4415	0.3584	0.2001	0.0000	0.0000
$\alpha_2 = 3$	*separable*	$N_p^s = 3$	3	0.5201	0.3952	0.0847	0.0000	0.0000
$\alpha_3 = 5$	*nonseparable*	$N_p = 4$	4	0.3795	0.3369	0.2607	0.0229	0.0000
$\alpha_3 = 5$	*separable*	$N_p^s = 3$	3	0.4454	0.3705	0.1841	0.0000	0.0000

Recall that contestants have a constant level of absolute prudence τ with CARA preferences and $\tau = \alpha$. In the case of $\alpha = 5$, the optimal number of prizes reaches its maximum, N_p or N_p^s, irrespective of nonseparable or separable utility. A positive prize for the 4th rank yields a direct negative incentive because $\mu_4 \approx -0.2833 < 0$: It is impossible under separable utility, whereas it could arise under nonseparable utility. The loss caused to marginal benefit of effort is compensated for by reduced marginal effort cost when contestants become sufficiently prudent.

5 Concluding Remarks

Both winner-take-all and multiple-prize contests are often observed in practice. A growing strand of literature has examined optimal prize allocation in various economic contexts (see Sisak, 2009). Clark and Riis (1998) and Schweinzer and Segev (2012) establish the winner-take-all principle for risk-neutral players within the framework of multi-winner nested Tullock contests. In this paper, we introduce risk aversion into this framework. We demonstrate that in contrast to the case of risk-neutral contestants, a variation in prize allocation affects both the marginal benefit and marginal cost of effort, and contestants' risk attitude reshapes the trade-off. The analysis provides sufficient conditions under which a multi-prize or a single-prize contest emerges in the optimum. It is shown that a multi-prize contest is more likely to outperform a single-prize one when contestants become more risk averse and more prudent. In particular, prudence plays a nontrivial role that allows additional prizes to reduce marginal effort cost, which makes a low-rank prize possible even if it creates direct negative incentive.

Our study leaves large room for future extension. In this paper, we abstract away contestants' ex ante participation decisions and focus on a setting in which all contestants participate with zero entry cost. It would be interesting to endogenize contestants' entry with the presence of risk aversion. Intuitively, multiple prizes provide insurance against a contestant's income shock upon entry, and a natural conjecture is that multiple prizes would be more appealing when contestants bear an entry cost and decide whether to enter the contest in the first place. Another possible avenue for future research is to extend our analysis in the static model to a dynamic setting, in which contestants must endure a series of shots to advance toward the top (e.g., Rosen, 1986; Gradstein and Konrad, 1999; Fu and Lu, 2012a, among others). The hierarchical winner-take-all principle—which requires that the entire prize purse be allocated to a single grand prize—deserves to be reexamined in an environment with risk-averse contestants. It would also be intriguing to explore whether our main results would persist under alternative or more general static winner selection mechanisms—for instance, the multi-prize "reverse" nested lottery contest proposed by Fu, Lu and Wang (2014). Finally, we model contestants' risk attitude by introducing risk aversion, while the impact of risk attitude can be examined from alternative perspectives, such as preferences with loss aversion (e.g., Chen, Ong and Segev, 2017). It is worthwhile to compare the implications of these two approaches on the optimal prize structures. We leave the exploration of these possibilities for future research.

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Appendix

Proof of Theorem 1

Proof. It is useful to state several intermediate results.

Lemma 1 ***(Fu, Wu and Zhu, 2019)*** *Suppose that Assumptions 1 and 3 are satisfied. Then* $\frac{\partial^2 \pi(e',e)}{\partial e'^2} \leq 0$ *for all* $r \in (0,1]$.

Lemma 2 *Suppose that Assumptions 1 and 3 are satisfied. Then*

$$\big[u(x) - u(y)\big] \times \big[u''(x) - u''(y)\big] \geq \big[u'(y) - u'(x)\big]^2, \textit{ for all } x > y.$$

Proof. Because $-u''(c)/u'(c)$ is nonincreasing in c, we have that $\frac{d}{dc}\left(-\frac{u''(c)}{u'(c)}\right) \leq 0$, which is equivalent to

$$u'(t)u'''(t) \geq \big[-u''(t)\big]^2. \tag{9}$$

This implies instantly that $u'''(\cdot) \geq 0$. Moreover, fixing $x > y$, we have that

$$\begin{aligned}
\big[u(x) - u(y)\big] \times \big[(u''(x) - u''(y)\big] &= \left(\int_y^x u'(t)dt\right)\left(\int_y^x u'''(t)dt\right) \\
&\geq \left(\int_y^x \sqrt{u'(t)u'''(t)}dt\right)^2 \\
&\geq \left(\int_y^x -u''(t)dt\right)^2 = \big[u'(y) - u'(x)\big]^2,
\end{aligned}$$

where the first inequality follows from Cauchy's inequality and the second inequality follows directly from (9). This completes the proof. ■

Now we can prove Theorem 1. Clearly, $\boldsymbol{e} = (0, \ldots, 0)$ cannot constitute an equilibrium; together with Lemma 1 and the first-order condition (3), it suffices to show that for all $r \leq \bar{r}$, there exists a unique solution to

$$F\left(e; \boldsymbol{V}\right) := r \times \sum_{m=1}^{N} \big[\mu_m \times u(w + V_m - e)\big] - e \times \sum_{m=1}^{N} u'(w + V_m - e) = 0.$$

Existence of Equilibrium We first show that $F(0; \boldsymbol{V}) > 0$. Note that μ_m is strictly decreasing in m and $\sum_{m=1}^{N} \mu_m = 0$. Define $\kappa := \max\{j = 1, \ldots, N \,|\, \mu_j \geq 0.\}$. It can be verified that κ is well defined, unique, and $\kappa \leq N-1$. Moreover, we have that

$$
\begin{aligned}
F(0;\boldsymbol{V}) = r \times \sum_{m=1}^{N} \big[\mu_m \times u(w+V_m)\big] &= r \times \left\{ \sum_{m=1}^{\kappa} \big[\mu_m \times u(w+V_m)\big] + \sum_{m=\kappa+1}^{N} \big[\mu_m \times u(w+V_m)\big] \right\} \\
&\geq r \times \left\{ \sum_{m=1}^{\kappa} \big[\mu_m \times u(w+V_\kappa)\big] + \sum_{m=\kappa+1}^{N} \big[\mu_m \times u(w+V_{\kappa+1})\big] \right\} \\
&= r \times \left(\sum_{m=1}^{\kappa} \mu_m \right) \times \big[u(w+V_\kappa) - u(w+V_{\kappa+1})\big] \geq 0.
\end{aligned}
$$

Note that the equal sign in the above two inequalities occurs simultaneously if and only if $V_1 = \ldots = V_N$, which is excluded by assumption. Therefore, we have that $F(0;\boldsymbol{V}) > 0$.

Next, we show that $F(e;\boldsymbol{V}) < 0$ for sufficiently large e. Note that $F(e;\boldsymbol{V})$ can be bounded above by

$$
\begin{aligned}
F(e;\boldsymbol{V}) :&= r \times \sum_{m=1}^{N} \big[\mu_m \times u(w+V_m-e)\big] - e \times \sum_{m=1}^{N} u'(w+V_m-e) \\
&= r \times \sum_{m=1}^{N-1} \left\{ \left(\sum_{j=1}^{m} \mu_j \right) \times \big[u(w+V_m-e) - u(w+V_{m+1}-e)\big] \right\} - e \times \sum_{m=1}^{N} u'(w+V_m-e) \\
&\leq r \times \sum_{m=1}^{N-1} \left\{ \left(\sum_{j=1}^{m} \mu_j \right) \times (V_m - V_{m+1}) \times u'(w+V_{m+1}-e) \right\} - e \times \sum_{m=1}^{N} u'(w+V_m-e) \\
&\leq \sum_{m=1}^{N-1} \big[rNV \times u'(w+V_{m+1}-e)\big] - e \times \sum_{m=1}^{N} u'(w+V_m-e) \\
&= -e \times u'(w+V_1-e) + (rNV-e) \times \sum_{m=2}^{N} u'(w+V_m-e),
\end{aligned}
$$

where the second equality follows from $\sum_{m=1}^{N} \mu_m = 0$; the first inequality follows from the concavity of $u(\cdot)$ and the fact that $\sum_{j=1}^{m} \mu_j \geq 0$ for all $m \in \{1,\ldots,N-1\}$; the second inequality follows from $V_m - V_{m+1} \leq V$ and $\sum_{j=1}^{m} \mu_j \leq N$. It is clear that the last term is negative if $e > rNV$. Therefore, there exists at least one equilibrium of the contest game.

Uniqueness of Equilibrium Next, we prove the uniqueness of equilibrium. It suffices to show that if $F(e_0;\boldsymbol{V}) = 0$ for some $e_0 > 0$, then we must have $\frac{\partial F(e_0;\boldsymbol{V})}{\partial e} < 0$ (see Treich, 2010). For notational convenience, let us denote $\sum_{j=m+1}^{n} \mu_j$ by $\widetilde{\mu}_m$ for all $m \in \{1,\ldots,N-1\}$. It is straightforward to verify that $\widetilde{\mu}_m < 0$. Then $F(e_0; \boldsymbol{V}) = 0$ can be rewritten as

$$\begin{aligned}
&\sum_{m=1}^{N-1}(N-m)\left[u'(w+V_{m+1}-e_0)-u'(w+V_m-e_0)\right]+Nu'(w+V_1-e_0)\\
=&\frac{r}{e_0}\times\sum_{m=1}^{N-1}\widetilde{\mu}_m\left[u(w+V_{m+1}-e_0)-u(w+V_m-e_0)\right]. \qquad (10)
\end{aligned}$$

Moreover, we have that

$$\begin{aligned}
\left.\frac{\partial F(e;\boldsymbol{V})}{\partial e}\right|_{e=e_0} =& -r\times\sum_{m=1}^{N-1}\widetilde{\mu}_m\left[u'(w+V_{m+1}-e_0)-u'(w+V_m-e_0)\right]\\
&+e_0\times\sum_{m=1}^{N-1}(N-m)\left[u''(w+V_{m+1}-e_0)-u''(w+V_m-e_0)\right]+Ne_0u''(w+V_1-e_0)\\
&-\left\{\sum_{m=1}^{N-1}(N-m)\left[u'(w+V_{m+1}-e_0)-u'(w+V_m-e_0)\right]+Nu'(w+V_1-e_0)\right\}\\
=& -r\times\sum_{m=1}^{N-1}\widetilde{\mu}_m\left[u'(w+V_{m+1}-e_0)-u'(w+V_m-e_0)\right]\\
&+e_0\times\sum_{m=1}^{N-1}(N-m)\left[u''(w+V_{m+1}-e_0)-u''(w+V_m-e_0)\right]\\
&-\frac{r}{e_0}\times\sum_{m=1}^{N-1}\widetilde{\mu}_m\left[u(w+V_{m+1}-e_0)-u(w+V_m-e_0)\right]+Ne_0u''(w+V_1-e_0)\\
\leq& -\sum_{m=1}^{N-1}\left\{\begin{array}{c}
-r\widetilde{\mu}_m\left[u'(w+V_m-e_0)-u'(w+V_{m+1}-e_0)\right]\\
+e_0(N-m)\left[u''(w+V_m-e_0)-u''(w+V_{m+1}-e_0)\right]\\
-\frac{r}{e_0}\widetilde{\mu}_m\left[u(w+V_m-e_0)-u(w+V_{m+1}-e_0)\right]
\end{array}\right\},
\end{aligned}$$

where the second equality follows directly from (10), and the inequality follows from $u'' \leq 0$. Therefore, it suffices to show that for all $m \in \{1, \ldots, N-1\}$ and $r \leq \bar{r}$, the following inequality holds:

$$\left[\begin{array}{c}
e_0(N-m)\left[u''(w+V_m-e_0)-u''(w+V_{m+1}-e_0)\right]\\
+\frac{r}{e_0}\left(-\widetilde{\mu}_m\right)\left[u(w+V_m-e_0)-u(w+V_{m+1}-e_0)\right]
\end{array}\right]\geq r\left(-\widetilde{\mu}_m\right)\left[\begin{array}{c}
u'(w+V_{m+1}-e_0)\\
-u'(w+V_m-e_0)
\end{array}\right].$$

Recall that $\widetilde{\mu}_m < 0$. Therefore, we have that

$$
\begin{aligned}
& e_0(N-m)\left[u''(w+V_m-e_0)-u''(w+V_{m+1}-e_0)\right]+\frac{r}{e_0}(-\widetilde{\mu}_m)\left[u(w+V_m-e_0)-u(w+V_{m+1}-e_0)\right] \\
\geq & 2\sqrt{r(N-m)(-\widetilde{\mu}_m)}\sqrt{\left[u''(w+V_m-e_0)-u''(w+V_{m+1}-e_0)\right]\times\left[u(w+V_m-e_0)-u(w+V_{m+1}-e_0)\right]} \\
\geq & 2\sqrt{r(N-m)(-\widetilde{\mu}_m)}\left[u'(w+V_{m+1}-e_0)-u'(w+V_m-e_0)\right],
\end{aligned}
$$

where the first inequality follows from the AM-GM inequality, and the second inequality follows from Lemma 2. To complete the proof of uniqueness of equilibrium, it suffices to show that

$$2\sqrt{r(N-m)(-\widetilde{\mu}_m)} \geq r\left(-\widetilde{\mu}_m\right),$$

which is equivalent to

$$r\left(-\widetilde{\mu}_m\right) \leq 4(N-m).$$

The above condition holds for all $m \in \{1,\ldots,N-1\}$ under Assumption 2. To see this more clearly, recall that

$$r \leq \bar{r} \equiv \min\left\{1, \frac{4}{\sum_{g=2}^{N}\frac{1}{g}}\right\} \leq \frac{4}{\sum_{g=2}^{N}\frac{1}{g}}.$$

This in turn implies that

$$
\begin{aligned}
r\left(-\widetilde{\mu}_m\right) \leq \frac{4}{\sum_{g=2}^{N}\frac{1}{g}}\left(-\widetilde{\mu}_m\right) &\equiv -\frac{4}{\sum_{g=2}^{N}\frac{1}{g}}\sum_{j=m+1}^{N}\mu_j \\
&\leq -\frac{4}{\sum_{g=2}^{N}\frac{1}{g}}\sum_{j=m+1}^{N}\mu_N \\
&= -\frac{4}{\sum_{g=2}^{N}\frac{1}{g}}(N-m)\mu_N = 4(N-m),
\end{aligned}
$$

where the second inequality follows from the fact that $\mu_m \geq \mu_N$ for all $m \in \mathcal{N}$, and the last equality follows from $\mu_N \equiv 1 - \sum_{g=0}^{N-1}\frac{1}{N-g} = -\sum_{g=2}^{N}\frac{1}{g}$. This concludes the proof. ■

Proof of Corollary 1

Proof. Exploiting the CARA functional form (i.e., $u(c) = 1 - \exp(-\alpha c)$, with $\alpha > 0$) and the first-order condition (3), we can solve for the unique equilibrium effort level as follows:

$$e = r \times \frac{\sum_{m=1}^{N}\left[\mu_m \times u(w+V_m-e)\right]}{\sum_{m=1}^{N} u'(w+V_m-e)} = \frac{r}{\alpha} \times \frac{\sum_{m=1}^{N}\left[\mu_m \times \exp(-\alpha V_m)\right]}{\sum_{m=1}^{N}\exp(-\alpha V_m)}.$$

The above observation, together with Theorem 1, implies instantly that exists a unique symmetric pure-strategy equilibrium of the contest game for all $r \leq 1$. This concludes the proof. ■

Proof of Theorem 2

Proof. For notational convenience, let us define $z := e^r$, $z' := (e')^r$, and

$$P_m^{\dagger}\left(z', z\right) := \frac{(N-1)!}{(N-m)!} \times \left(\prod_{j=1}^{m-1} \frac{z}{(N-j)z + z'}\right) \times \frac{z'}{(N-m)z + z'}.$$

It follows immediately that $P_m\left(e', e\right) = P_m^{\dagger}\left(z', z\right)$. Next, let us define

$$\widetilde{P}_m(e', e) := \sum_{j=1}^{m} P_j(e', e).$$

In words, $\widetilde{P}_m(e', e)$ is an indicative contestant's probability of obtaining the first m prizes when he exerts effort e' and all other contestants exert effort e. An indicative contestant's expected payoff [i.e., Equation (2)] can then be rewritten as

$$\pi(e', e) = \sum_{m=1}^{N-1} \widetilde{P}_m(e', e)\left[u(w + V_m - e') - u(w + V_{m+1} - e')\right] + u(w + V_N - e'). \tag{11}$$

Similarly, we can define $\widetilde{P}_m^{\dagger}(z', z)$ as

$$\widetilde{P}_m^{\dagger}(z', z) := \sum_{j=1}^{m} P_j^{\dagger}(z', z).$$

It can be verified that

$$1 - \widetilde{P}_m^{\dagger}(z', z) = \frac{N!}{(N-m)!} \times \left(\prod_{j=0}^{m-1} \frac{z}{z' + (N-1)z - jz}\right). \tag{12}$$

The following lemmata can be obtained:

Lemma 3 ***(Fu, Wu and Zhu, 2019)*** $\frac{\partial^2 \widetilde{P}_m^{\dagger}(z', z)}{\partial z'^2} \leq 0$ *for all* $m \in \mathcal{N}$.

Lemma 4 *Suppose that* $r \leq 1$. *Then* $\frac{\partial^2 \widetilde{P}_m(e', e)}{\partial e'^2} \leq 0$ *for all* $m \in \mathcal{N}$.

Proof. Carrying out the algebra, we have that

$$\frac{\partial^2 \widetilde{P}_m(e',e)}{\partial e'^2} = \frac{\partial^2 \widetilde{P}_m^{\dagger}(z',z)}{\partial z'^2} r^2 (e')^{2r-2} + \frac{\partial \widetilde{P}_m^{\dagger}(z',z)}{\partial z'} r(r-1)(e')^{r-2} \leq 0,$$

where the inequality follows from Lemma 3 and $r \in (0,1]$. This competes the proof. ■

Now we can prove Theorem 2. Without any loss of generality, we can suppose that contestants' utility function is $u(c) = c - \frac{\gamma}{2}c^2$, with $\gamma > 0$. Note that $u'(w+V) \geq 0$ implies $w + V \leq 1/\gamma$. It is straightforward to verify that the first-order condition (3) has a unique positive solution with quadratic utility functions. From the proof of Theorem 1, it suffices to show that $\frac{\partial^2 \pi(e',e)}{\partial e'^2} \leq 0$ for all $r \leq \frac{1}{2(N-1)^2+1}$.

For notational convenience, let us define $d_m := V_m - V_{m+1} \geq 0$, with $m \in \{1, \ldots, N-1\}$. It follows from Equation (11) that

$$\begin{aligned}
\frac{\partial^2 \pi(e',e)}{\partial e'^2} =& \sum_{m=1}^{N-1} \left\{ d_m \frac{\partial^2 \widetilde{P}_m(e',e)}{\partial e'^2} \left[1 - \frac{\gamma}{2}\left(2w + V_{m+1} + V_m - 2e'\right) \right] + 2\gamma \frac{\partial \widetilde{P}_m(e',e)}{\partial e'} d_m \right\} - \gamma \\
\leq& \sum_{m=1}^{N-1} \left[d_m \frac{\partial^2 \widetilde{P}_m(e',e)}{\partial e'^2} \left\{ 1 - \frac{\gamma}{2} \left[V_{m+1} + V_m - 2\left(V_1 - \frac{1}{\gamma}\right) \right] \right\} + 2\gamma \frac{\partial \widetilde{P}_m(e',e)}{\partial e'} d_m \right] - \gamma \\
=& \frac{\gamma}{2} \times \left\{ \sum_{m=1}^{N-1} \left[\frac{\partial^2 \widetilde{P}_m(e',e)}{\partial e'^2} d_m \left(2\sum_{j=1}^{m-1} d_j + d_m \right) + 4 \frac{\partial \widetilde{P}_m(e',e)}{\partial e'} d_m \right] - 2 \right\} \\
\leq& \frac{\gamma}{2} \times \sum_{m=1}^{N-1} \left[\frac{\partial^2 \widetilde{P}_m(e',e)}{\partial e'^2} d_m^2 + 4 \frac{\partial \widetilde{P}_m(e',e)}{\partial e'} d_m - \frac{2}{N-1} \right],
\end{aligned}$$

where the first inequality follows from Lemma 4 and $w - e' \leq \frac{1}{\gamma} - V_1$; and the second inequality follows again from Lemma 4 and $d_m \geq 0$. Therefore, it suffices to show that for all $m \in \{1, \ldots, N-1\}$,

$$\frac{\partial^2 \widetilde{P}_m(e',e)}{\partial e'^2} d_m^2 + 4 \frac{\partial \widetilde{P}_m(e',e)}{\partial e'} d_m - \frac{2}{N-1} \leq 0.$$

View the left-hand side of the above inequality as a function of d_m. Note that $\frac{\partial^2 P_m(e',e)}{\partial e'^2} \leq 0$ by Lemma 4. Therefore, a sufficient condition for the above inequality to hold is

$$2\left[\frac{\partial \widetilde{P}_m(e',e)}{\partial e'}\right]^2 + \frac{1}{N-1} \times \frac{\partial^2 \widetilde{P}_m(e',e)}{\partial e'^2} \leq 0. \tag{13}$$

Note that

$$\frac{\partial \widetilde{P}_m(e',e)}{\partial e'} = \frac{\partial \widetilde{P}_m^\dagger(z',z)}{\partial z'} r(e')^{r-1}, \tag{14}$$

and

$$\begin{aligned}\frac{\partial^2 \widetilde{P}_m(e',e)}{\partial e'^2} &= \frac{\partial^2 \widetilde{P}_m^\dagger(z',z)}{\partial z'^2} r^2(e')^{2r-2} + \frac{\partial \widetilde{P}_m^\dagger(z',z)}{\partial z'} r(r-1)(e')^{r-2} \\ &\leq -\frac{\partial \widetilde{P}_m^\dagger(z',z)}{\partial z'} r(1-r)(e')^{r-2},\end{aligned} \tag{15}$$

where the inequality follows from Lemma 3. Combining (13), (14), and (15), it suffices to show that

$$z'\frac{\partial \widetilde{P}_m^\dagger(z',z)}{\partial z'} \leq \frac{1-r}{2r(N-1)}.$$

Next, note that the right-hand side of the above inequality can be bounded above by

$$\begin{aligned}z'\frac{\partial \widetilde{P}_m^\dagger(z',z)}{\partial z'} =& z'\widetilde{P}_m^\dagger(z',z)\frac{\partial \ln \widetilde{P}_m^\dagger(z',z)}{\partial z'} \\ =& z'\widetilde{P}_m^\dagger(z',z)\left(-\frac{\partial \ln\left(1-\widetilde{P}_m^\dagger(z',z)\right)}{\partial z'}\right) \\ =& z'\widetilde{P}_m^\dagger(z',z)\left[-\frac{\partial}{\partial z'}\left(\ln(N!) - \ln(N-m)! + m\ln z - \sum_{j=0}^{m-1}\ln\left(z' + (N-1)z - jz\right)\right)\right] \\ =& \widetilde{P}_m^\dagger(z',z)\sum_{j=0}^{m-1}\frac{z'}{z'+(N-1)z-jz} \leq m \leq N-1,\end{aligned}$$

where the third equality follows from (12); and the first inequality follows from the fact that $\widetilde{P}_m^\dagger(z',z) \leq 1$ and $\frac{z'}{z'+(N-1)z-jz} \leq 1$. The above inequality, together with $r \leq \frac{1}{2(N-1)^2+1}$, implies instantly that

$$z'\frac{\partial \widetilde{P}_m^\dagger(z',z)}{\partial z'} \leq N-1 \leq \frac{1-r}{2r(N-1)}.$$

This completes the proof. ■

Proof of Proposition 1

Proof. Denote the optimal prize allocation by $\boldsymbol{V}^* := (V_1^*, \ldots, V_N^*)$ and the equilibrium effort level by e^*. Suppose to the contrary that there exists some $j \in \{2, \ldots, N\}$ such that $V_j^* = V_{j-1}^*$. Consider an alternative prize structure $\boldsymbol{V}^\dagger := (V_1^\dagger, \ldots, V_N^\dagger)$, with $V_m^\dagger = V_m^*$ for

$m \in \mathcal{N} \setminus \{j-1, j\}$, $V_{j-1}^{\dagger} = V_{j-1}^{*} + \epsilon$, and $V_j^{\dagger} = V_j^{*} - \epsilon$. Next, we show that the equilibrium effort under $\boldsymbol{V}^{\dagger}$ is greater than that under $\boldsymbol{V}^{*}$ for an infinitesimal $\epsilon > 0$. It suffices to show that $H(\epsilon) := F(e^*; \boldsymbol{V}^{\dagger}) - F(e^*; \boldsymbol{V}^{*}) > 0$. Simple algebra yields that

$$
\begin{aligned}
H(\epsilon) =& r \times \left\{ \mu_{j-1} \Big[u(w + V_j^* + \epsilon - e^*) - u(w + V_j^* - e^*) \Big] + \mu_j \Big[u(w + V_j^* - \epsilon - e^*) - u(w + V_j^* - e^*) \Big] \right\} \\
&- e^* \times \left\{ \Big[u'(w + V_j^* + \epsilon - e^*) - u'(w + V_j^* - e^*) \Big] + \Big[u'(w + V_j^* - \epsilon - e^*) - u'(w + V_j^* - e^*) \Big] \right\}.
\end{aligned}
$$

Clearly, $H(0) = 0$. Moreover, we have

$$
H'(0) = r \times \big(\mu_{j-1} - \mu_j \big) \times u'(w + V_j^* - e^*) > 0.
$$

Therefore, $H(\epsilon) > 0$ for sufficiently small $\epsilon > 0$. This completes the proof. ■

Proof of Proposition 2

Proof. Recall that the optimal prize allocation and equilibrium effort level are denoted by $\boldsymbol{V}^* := (V_1^*, \ldots, V_N^*)$ and e^* in the proof of Proposition 1, respectively. Suppose to the contrary that a contest with multiple prizes is optimal. Then there exists a prize $j \in \{2, \ldots, N\}$ such that $V_j^* > 0$. Consider the following alternative prize allocation, denoted by $\widehat{\boldsymbol{V}}$, that decreases V_j^* by a small amount $\epsilon > 0$, and increases V_1^* by the same amount. Next, we show that the constructed prize allocation generates more revenue to the contest designer. By the same argument as in the proof of Proposition 1, it suffices to show that $\zeta(\epsilon) := F(e^*; \widehat{\boldsymbol{V}}) - F(e^*; \boldsymbol{V}^*) > 0$. Clearly, $\zeta(0) = 0$. Carrying out the algebra, we have that

$$
\begin{aligned}
\zeta(\epsilon) =& r \times \left\{ \mu_1 \left[u(w + V_1^* + \epsilon - e^*) - u(w + V_1^* - e^*) \right] + \mu_j \Big[u(w + V_j^* - \epsilon - e^*) - u(w + V_j^* - e^*) \Big] \right\} \\
&- e^* \times \left\{ \Big[u'(w + V_1^* + \epsilon - e^*) - u'(w + \widehat{V}_1 - e^*) \Big] + \Big[u'(w + V_j^* - \epsilon - e^*) - u'(w + V_j^* - e^*) \Big] \right\},
\end{aligned}
$$

and thus

$$
\begin{aligned}
\zeta'(0) =& r \times \Big[\mu_1 u'(w + V_1^* - e^*) - \mu_j u'(w + V_j^* - e^*) \Big] \\
&- e^* \times \Big[u''(w + V_1^* - e^*) - u''(w + V_j^* - e^*) \Big] \qquad (16)
\end{aligned}
$$

By Proposition 1, we have that $V_1^* > V_j^*$; together with $-u'''/u'' \leq \tau$, we have $u''' \leq -\tau u''$

and thus

$$
\begin{aligned}
u''(w+V_1^*-e^*)-u''(w+V_j^*-e^*) &= \int_{w+V_j^*-e^*}^{w+V_1^*-e^*} u'''(e)de \\
&\leq -\tau \int_{w+V_j^*-e^*}^{w+V_1^*-e^*} u''(e)de \\
&= -\tau \Big[u'(w+V_1^*-e^*)-u'(w+V_j^*-e^*)\Big]. \qquad (17)
\end{aligned}
$$

Combing (16) and (17) yields that

$$
\begin{aligned}
\zeta'(0) &\geq (r\mu_1+e^*\tau)\times u'(w+V_1^*-e^*)-(r\mu_j+e^*\tau)\times u'(w+V_j^*-e^*) \\
&\geq (r\mu_1+e^*\tau)\times u'(w+V_1^*-e^*)-(r\mu_2+e^*\tau)\times u'(w+V_j^*-e^*) \\
&= (r\mu_2+e^*\tau)\times u'(w+V_1^*-e^*)\times\left[\frac{r\mu_1+e^*\tau}{r\mu_2+e^*\tau}-\frac{u'(w+V_j^*-e^*)}{u'(w+V_1^*-e^*)}\right] \\
&> (r\mu_2+e^*\tau)\times u'(w+V_1^*-e^*)\times\left[\frac{r\mu_1+e^*\tau}{r\mu_2+e^*\tau}-\frac{u'(w-e^*)}{u'(w+V-e^*)}\right] \\
&> (r\mu_2+e^*\tau)\times u'(w+V_1^*-e^*)\times\left[\frac{r\mu_1+\frac{V}{N}\tau}{r\mu_2+\frac{V}{N}\tau}-\frac{u'(w-\frac{V}{N})}{u'(w+V)}\right] \geq 0,
\end{aligned}
$$

where the second inequality follows from $\mu_2 \geq \mu_j$ for $j \geq 2$ and $u' > 0$; the third inequality follows from $u'' \leq 0$, $V_1^* < V$, and $V_j^* > 0$; the fourth inequality follows from $e^* \in (0, \frac{V}{N})$; and the last inequality follows from the condition assumed in Proposition 2. This completes the proof. ■

Proof of Proposition 3

Proof. Note that when $N \geq 3$, we have $\mu_1 > \mu_2 > 0$. Now consider introducing a small second prize $\epsilon > 0$, and denote the corresponding prize structure by $\widetilde{\boldsymbol{V}} := (\widetilde{V}_1, \ldots, \widetilde{V}_N) = (V-\epsilon, \epsilon, 0, \ldots, 0)$. Following the same argument as in the proof of Proposition 1, it suffices to show that $G(\epsilon) := F(e_s; \widetilde{\boldsymbol{V}}) - F(e_s; \widetilde{\boldsymbol{V}}) > 0$. Carrying out the algebra, we have that

$$
\begin{aligned}
G(\epsilon) =& r\times\Big\{\mu_1\big[u(w+V-\epsilon-e_s)-u(w+V-e_s)\big]+\mu_2\big[u(w+\epsilon-e_s)-u(w-e_s)\big]\Big\} \\
&-e_s\times\Big\{\big[u'(w+V-\epsilon-e_s)-u'(w+V-e_s)\big]+\big[u'(w+\epsilon-e_s)-u'(w-e_s)\big]\Big\}.
\end{aligned}
$$

It is straightforward to see that $G(0) = 0$. Moreover, we have

$$\begin{aligned} G'(0) &= r \times \left[-\mu_1 u'(w+V-e_s) + \mu_2 u'(w-e_s)\right] + e_s \times \left[u''(w+V-e_s) - u''(w-e_s)\right] \\ &\geq r \times \left[-\mu_1 u'(w+V-e_s) + \mu_2 u'(w-e_s)\right] \\ &= r\mu_1 \times u'(w+V-e_s) \times \left[\frac{\mu_2}{\mu_1} \times \frac{u'(w-e_s)}{u'(w+V-e_s)} - 1\right] > 0, \end{aligned}$$

where the first inequality follows from $u''' \geq 0$, the second inequality follows from $\frac{u'(w-e_s)}{u'(w+V-e_s)} > \frac{\mu_1}{\mu_2}$. This completes the proof. ■

Proof of Corollary 2

Proof. Note that the equilibrium effort level for any prize allocation must be less than $\frac{V}{N}$. Therefore, $e_s < \frac{V}{N}$. Moreover, we have that

$$\frac{u'(w-e_s)}{u'(w+V-e_s)} \geq \frac{u'(w)}{u'\left(w + \frac{N-1}{N}V\right)},$$

and

$$\frac{\mu_1}{\mu_2} = \frac{1 - \frac{1}{N}}{1 - \frac{1}{N} - \frac{1}{N-1}}.$$

First, note that $\frac{u'(w)}{u'\left(w+\frac{N-1}{N}V\right)}$ is strictly increasing in V and is approaching $u'(w)/u'(\infty)$ as $V \nearrow \infty$. Moreover, the ratio $\frac{\mu_1}{\mu_2}$ is independent of V. Therefore, if $u'(\infty) < \frac{\mu_2}{\mu_1 u'(w)}$, then $\frac{u'(w-e_s)}{u'(w+V-e_s)} > \frac{\mu_1}{\mu_2}$ holds for sufficiently large V.

Second, note that $\frac{u'(w)}{u'\left(w+\frac{N-1}{N}V\right)}$ is strictly increasing in N and is approaching $\frac{u'(w)}{u'(w+V)} > 1$ as $N \nearrow \infty$; and $\frac{1-\frac{1}{N}}{1-\frac{1}{N}-\frac{1}{N-1}}$ is strictly decreasing in N and is approaching 1 as $N \nearrow \infty$. Therefore, $\frac{u'(w-e_s)}{u'(w+V-e_s)} > \frac{\mu_1}{\mu_2}$ as N becomes sufficiently large. This completes the proof. ■

Proof of Proposition 4

Proof. Suppose to the contrary that there exists a prize indexed by $k > N_p$ such that $V_k^* > 0$. By the definition of N_p, we must have

$$r \times \mu_k + \frac{V}{N}\tau \leq 0. \tag{18}$$

Now consider an alternative prize structure that decreases V_k^* and increases V_1^* by a small amount $\epsilon > 0$. Denote the constructed prize allocation by $\boldsymbol{V}^{\dagger\dagger}$. Next, we show that the equilibrium effort level under $\boldsymbol{V}^{\dagger\dagger}$ is greater than that under $\boldsymbol{V}^*$. Again, it suffices to show that $\chi(\epsilon) := F(e^*; \boldsymbol{V}^{\dagger\dagger}) - F(e^*; \boldsymbol{V}^*) > 0$ for sufficiently small $\epsilon > 0$. It can be verified that $\chi(0) = 0$. Carrying out the algebra, we have that

$$\begin{aligned}\chi(\epsilon) =& r \times \left\{\mu_1 \left[u(w + V_1^* + \epsilon - e^*) - u(w + V_1^* - e^*)\right] + \mu_k \left[u(w + V_k^* - \epsilon - e^*) - u(w + V_k^* - e^*)\right]\right\} \\ & - e^* \times \left\{\left[u'(w + V_1^* + \epsilon - e^*) - u'(w + V_1^* - e^*)\right] + \left[u'(w + V_k^* - \epsilon - e^*) - u'(w + V_k^* - e^*)\right]\right\},\end{aligned}$$

and

$$\chi'(0) = r \times \left[\mu_1 u'(w + V_1^* - e^*) - \mu_k u'(w + V_k^* - e^*)\right] - e^* \times \left[u''(w + V_1^* - e^*) - u''(w + V_k^* - e^*)\right] \tag{19}$$

Note that $-\frac{u'''(c)}{u''(c)} \leq \tau$ implies

$$u''(w + V_1^* - e^*) - u''(w + V_k^* - e^*) \leq -\tau \left[u'(w + V_1^* - e^*) - u'(w + V_k^* - e^*)\right]. \tag{20}$$

Combing (19) and (20) yields that

$$\begin{aligned}\chi'(0) &\geq (r\mu_1 + e^*\tau) \times u'(w + V_1^* - e^*) - (r\mu_k + e^*\tau) \times u'(w + V_k^* - e^*) \\ &\geq (r\mu_1 + e^*\tau) \times u'(w + V_1^* - e^*) - (r\mu_k + \frac{V}{N}\tau) \times u'(w + V_k^* - e^*) \\ &\geq (r\mu_1 + e^*\tau) \times u'(w + V_1^* - e^*) > 0,\end{aligned}$$

where the second inequality follows from $e^* \leq \frac{V}{N}$, and the third inequality follows from (18) and $u' > 0$. This completes the proof. ■

Proof of Corollary 3

Proof. Note that $u''' = 0$ if the utility function is quadratic. The proof for the optimality of multiple prizes follows immediately from Proposition 3, and it remains to show that it is optimal for the contest designer to award a single prize if $\frac{u'(w-e_s)}{u'(w+V-e_s)} < \frac{\mu_1}{\mu_2}$. It is useful to prove an intermediate result.

Lemma 5 *Suppose that $u(\cdot)$ is strictly increasing, concave, continuously differentiable, and exhibits increasing absolute risk aversion (IARA, henceforth). Then*

$$\frac{u'(c_1)}{u'(c_1 + \Delta)} < \frac{u'(c_2)}{u'(c_2 + \Delta)}, \text{ for all } \Delta > 0 \text{ and } c_1 < c_2.$$

Proof. It suffices to show that $d\left[\frac{u'(c)}{u'(c+\Delta)}\right]/dc > 0$. Note that

$$\begin{aligned}\frac{d}{dc}\left[\frac{u'(c)}{u'(c+\Delta)}\right] &= \frac{u''(c)u'(c+\Delta)-u'(c)u''(c+\Delta)}{\left[u'(c+\Delta)\right]^2}\\ &= \frac{u'(c)}{u'(c+\Delta)}\times\left\{\left[-\frac{u''(c+\Delta)}{u'(c+\Delta)}\right]-\left[-\frac{u''(c)}{u'(c)}\right]\right\}>0,\end{aligned}$$

where the strict inequality follows directly from the IARA assumption on the utility function. This completes the proof. ■

We are now ready to prove Corollary 3. Suppose to the contrary that

$$\frac{u'(w-e_s)}{u'(w+V-e_s)} < \frac{\mu_1}{\mu_2}, \tag{21}$$

and awarding multiple prizes is optimal. We follow the notation in the proof of Proposition 2 and denote the optimal prize structure and the corresponding equilibrium effort by $\boldsymbol{V}^* \equiv (V_1^*, \ldots, V_N^*)$ and e^*, respectively. Because awarding multiple prizes is optimal by assumption, it follows immediately from Proposition 1 that $V_1^* > V_2^* > 0$ and $e^* \geq e_s$. Consider an alternative prize allocation that decreases V_2^* by a small amount $\epsilon > 0$, and increases V_1^* by the same amount. Next, we show this alternative allocation generates more effort than $\boldsymbol{V}^*$. Applying the same argument as in the proof of Proposition 2, it suffices to show that

$$\begin{aligned}\zeta'(0) = r \times &\left[\mu_1 u'(w+V_1^*-e^*) - \mu_2 u'(w+V_2^*-e^*)\right]\\ &- e^* \times \left[u''(w+V_1^*-e^*) - u''(w+V_2^*-e^*)\right] > 0.\end{aligned}$$

Note that $u''(c)$ is a constant due to the assumption of quadratic utility function. Therefore, $\zeta'(0) > 0$ is equivalent to

$$\frac{u'(w+V_2^*-e^*)}{u'(w+V_1^*-e^*)} < \frac{\mu_1}{\mu_2}.$$

Note that

$$\frac{u'(w+V_2^*-e^*)}{u'(w+V_1^*-e^*)} < \frac{u'(w-e^*)}{u'(w+V-e^*)} \leq \frac{u'(w-e_s)}{u'(w+V-e_s)} < \frac{\mu_1}{\mu_2},$$

where the first inequality follows from $0 < V_2^* < V_1^* < V$ and the strict concavity of $u(\cdot)$; the second inequality follows from $e^* \geq e_s$, the fact that a quadratic utility function exhibits IARA, and Lemma 5; and the last inequality follows directly from (21). This concludes the proof. ■

Proof of Corollary 4

Proof. By the same argument as in the proof of Proposition 3, it suffices to show that

$$G'(0) = r \times \big[-\mu_1 u'(w + V - e_s) + \mu_2 u'(w - e_s)\big] + e_s \times \big[u''(w + V - e_s) - u''(w - e_s)\big] > 0.$$

The first-order condition (3), together with the CARA functional form of utility $u(\cdot)$, implies that

$$\begin{aligned} e_s &= \frac{\sum_{m=1}^{N} \big[\mu_m \times u(w + V_m - e_s)\big]}{\sum_{m=1}^{N} u'(w + V_m - e_s)} r \\ &= -\frac{r}{\alpha} \times \frac{\sum_{m=1}^{N} \big[\mu_m \times \exp(-\alpha V_m)\big]}{\sum_{m=1}^{N} \exp(-\alpha V_m)} \\ &= \frac{r\mu_1}{\alpha} \times \frac{1 - \exp(-\alpha V)}{1 + \exp(-\alpha V)}, \end{aligned}$$

where the last equality follows from the postulated $\boldsymbol{V}_s \equiv (V, 0, \dots, 0)$. Note that $u''(c) = -\alpha \cdot u'(c)$, and thus $G'(0)$ can be simplified as

$$G'(0) = -(r\mu_1 + \alpha e_s) u'(w + V - e_s) + (r\mu_2 + \alpha e_s) u'(w - e_s).$$

Therefore, $G'(0) > 0$ is equivalent to

$$\frac{u'(w - e_s)}{u'(w + V - e_s)} > \frac{r\mu_1 + \alpha e_s}{r\mu_2 + \alpha e_s} = \frac{2\mu_1}{\mu_1 \big[1 - \exp(-\alpha V)\big] + \mu_2 \big[1 + \exp(-\alpha V)\big]}.$$

This completes the proof. ■

Proof of Proposition 5

Proof. With separable utility functions, denote the equilibrium effort under an arbitrary prize schedule $\boldsymbol{V} \equiv (V_1, \dots, V_N)$ by $e^s(\boldsymbol{V})$. It follows from (7) that

$$e^s(\boldsymbol{V}) = \frac{r}{N} \times \sum_{m=1}^{N} \big[\mu_m \times u(w + V_m)\big].$$

Suppose there exists some $j \in \mathcal{N}/\{1\}$ such that $\mu_j \leq 0$ and $V_j > 0$. Consider an alternative prize structure $\widehat{\boldsymbol{V}}^s := (\widehat{V}_1^s, \dots, \widehat{V}_N^s)$, with $\widehat{V}_m^s = V_m$ for $m \in \mathcal{N} \setminus \{1, j\}$, $\widehat{V}_1^s = V_1 + V_j$, and $\widehat{V}_j^s = 0$. It remains to show that the equilibrium effort under $\widehat{\boldsymbol{V}}^s$ is greater than that under $\boldsymbol{V}$. Carrying out the algebra, we have that

$$e^s(\widehat{\boldsymbol{V}}^s) - e^s(\boldsymbol{V}) = \frac{r}{N} \Big\{\mu_1 \big[u(w + V_1 + V_j) - u(w + V_1)\big] + \mu_j \big[u(w) - u(w + V_j)\big]\Big\} > 0,$$

where the strict inequality follows from $\mu_1 > 0 \geq \mu_j$, $V_1 + V_j > V_1 \geq V_j > 0$. This completes the proof. $\blacksquare$